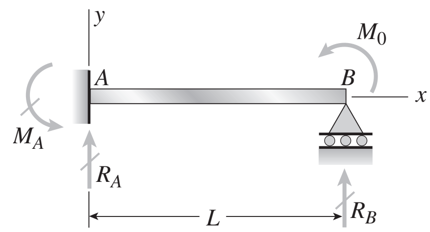
10

Statically Indeterminate Beams

Differential Equations of the Deflection Curve

The problems for Section 10.3 are to be solved by integrating the differential equations of the deflection curve. All beams have constant flexural rigidity EI. When drawing shear-force and bending-moment diagrams, be sure to label all critical ordinates, including maximum and minimum values.



Problem 10.3-1 A propped cantilever beam *AB* of length *L* is loaded by a counterclockwise moment M_0 acting at support *B* (see figure).

Beginning with the second-order differential equation of the deflection curve (the bending-moment equation), obtain the reactions, shear forces, bending moments, slopes, and deflections of the beam. Construct the shear-force and bending-moment diagrams, labeling all critical ordinates.

Solution 10.3-1 Propped cantilever beam

M_0 = applied load

Select M_A as the redundant reaction.

REACTIONS (FROM EQUILIBRIUM)

$$R_A = \frac{M_A}{L} + \frac{M_0}{L} \quad (1) \qquad R_B = -R_A \quad (2)$$

BENDING MOMENT (FROM EQUILIBRIUM)

$$M = R_A x - M_A = \frac{M_A}{L}(x - L) + \frac{M_0 x}{L} \quad (3)$$

DIFFERENTIAL EQUATIONS

$$EIv'' = M = \frac{M_A}{L}(x - L) + \frac{M_0 x}{L}$$

$$EIv' = \frac{M_A}{L}\left(\frac{x^2}{2} - Lx\right) + \frac{M_0 x^2}{2L} + C_1 \quad (4)$$

B.C. 1 $\quad v'(0) = 0 \qquad \therefore C_1 = 0$

$$EIv = \frac{M_A}{L}\left(\frac{x^3}{6} - \frac{Lx^2}{2}\right) + \frac{M_0 x^3}{6L} + C_2 \quad (5)$$

B.C. 2 $\quad v(0) = 0 \qquad \therefore C_2 = 0$

B.C. 3 $\quad v(L) = 0 \qquad \therefore M_A = \frac{M_0}{2}$

REACTIONS (SEE EQS. 1 AND 2)

$$M_A = \frac{M_0}{2} \qquad R_A = \frac{3M_0}{2L} \qquad R_B = -\frac{3M_0}{2L} \quad \longleftarrow$$

SHEAR FORCE (FROM EQUILIBRIUM)

$$V = R_A = \frac{3M_0}{2L} \quad \longleftarrow$$

BENDING MOMENT (FROM EQ. 3)

$$M = \frac{2M_0}{2L}(3x - L) \quad \longleftarrow$$

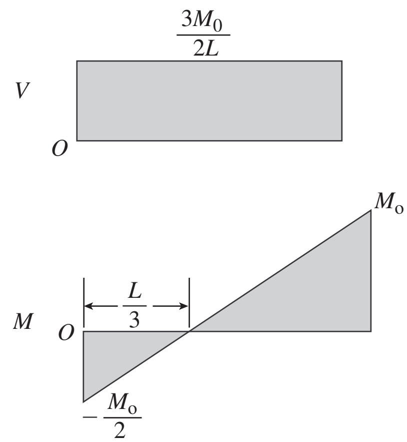
SLOPE (FROM EQ. 4)

$$v' = -\frac{M_0 x}{4LEI}(2L - 3x) \quad \longleftarrow$$

DEFLECTION (FROM EQ. 5)

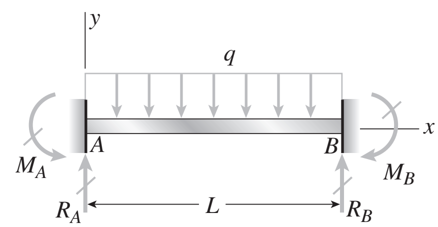
$$v = -\frac{M_0 x^2}{4LEI}(L - x) \quad \longleftarrow$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS



Problem 10.3-2 A fixed-end beam AB of length L supports a uniform load of intensity q (see figure).

Beginning with the second-order differential equation of the deflection curve (the bending-moment equation), obtain the reactions, shear forces, bending moments, slopes, and deflections of the beam. Construct the shear-force and bending-moment diagrams, labeling all critical ordinates.



Solution 10.3-2 Fixed-end beam (uniform load)

Select M_A as the redundant reaction.

REACTIONS (FROM SYMMETRY AND EQUILIBRIUM)

$$R_A = R_B = \frac{qL}{2} \qquad M_B = M_A$$

BENDING MOMENT (FROM EQUILIBRIUM)

$$M = R_A x - M_A - \frac{qx^2}{2} = -M_A + \frac{q}{2}(Lx - x^2) \qquad (1)$$

DIFFERENTIAL EQUATIONS

$$EIv'' = M = -M_A + \frac{q}{2}(Lx - x^2)$$

$$EIv' = -M_A x + \frac{q}{2}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) + C_1 \qquad (2)$$

B.C. 1 $v'(0) = 0 \quad \therefore C_1 = 0$

$$EIv = -\frac{M_A x^2}{2} + \frac{q}{2}\left(\frac{Lx^3}{6} - \frac{x^4}{12}\right) + C_2 \qquad (3)$$

B.C. 2 $v(0) = 0 \quad \therefore C_2 = 0$

B.C. 3 $v(L) = 0 \quad \therefore M_A = \dfrac{qL^2}{12}$

REACTIONS

$$R_A = R_B = \frac{qL}{2} \quad M_A = M_B = \frac{qL^2}{12} \quad \longleftarrow$$

SHEAR FORCE (FROM EQUILIBRIUM)

$$V = R_A - qx = \frac{q}{2}(L - 2x) \quad \longleftarrow$$

BENDING MOMENT (FROM EQ. 1)

$$M = -\frac{q}{12}(L^2 - 6Lx + 6x^2) \quad \longleftarrow$$

SLOPE (FROM EQ. 2)

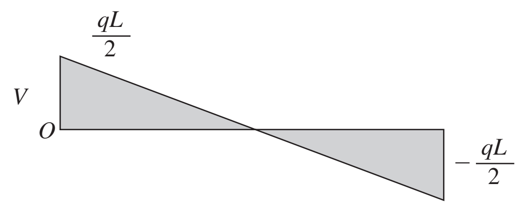
$$v' = -\frac{qx}{12EI}(L^2 - 3Lx + 2x^2) \quad \longleftarrow$$

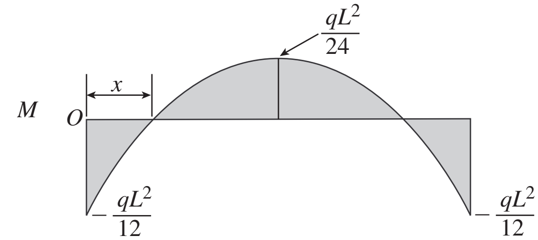
DEFLECTION (FROM EQ. 3)

$$v = -\frac{qx^2}{24EI}(L - x)^2 \quad \longleftarrow$$

$$\delta_{max} = -v\left(\frac{L}{2}\right) = \frac{qL^4}{384EI}$$

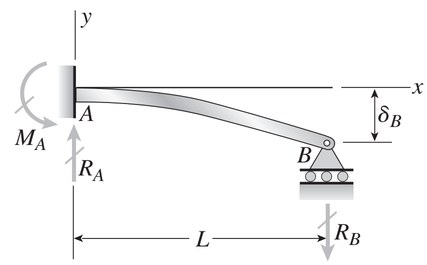
SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS





Problem 10.3-3 A cantilever beam AB of length L has a fixed support at A and a roller support at B (see figure). The support at B is moved downward through a distance δ_B.

Using the fourth-order differential equation of the deflection curve (the load equation), determine the reactions of the beam and the equation of the deflection curve. (*Note:* Express all results in terms of the imposed displacement δ_B.)



Solution 10.3-3 **Cantilever beam with imposed displacement δ_B**

REACTIONS (FROM EQUILIBRIUM)

$R_A = R_B$ (1) $M_A = R_B L$ (2)

DIFFERENTIAL EQUATIONS

$EIv'''' = -q = 0$ (3)

$EIv''' = V = C_1$ (4)

$EIv'' = M = C_1 x + C_2$ (5)

$EIv' = C_1 x^2/2 + C_2 x + C_3$ (6)

$EIv = C_1 x^3/6 + C_2 x^2/2 + C_3 x + C_4$ (7)

B.C. 1 $v(0) = 0$ $\therefore C_4 = 0$

B.C. 2 $v'(0) = 0$ $\therefore C_3 = 0$

B.C. 3 $v''(L) = 0$ $\therefore C_1 L + C_2 = 0$ (8)

B.C. 4 $v(L) = -\delta_B$ $\therefore C_1 L + 3C_2 = -6EI\delta_B/L^2$ (9)

SOLVE EQUATIONS (8) AND (9):

$$C_1 = \frac{3EI\delta_B}{L^3} \qquad C_2 = -\frac{3EI\delta_B}{L^2}$$

SHEAR FORCE (EQ. 4)

$$V = \frac{3EI\delta_B}{L^3} \qquad R_A = V(0) = \frac{3EI\delta_B}{L^3}$$

REACTIONS (EQS. 1 AND 2)

$$R_A = R_B = \frac{3EI\delta_B}{L^3} \qquad M_A = R_B L = \frac{3EI\delta_B}{L^2} \quad \longleftarrow$$

DEFLECTION (FROM EQ. 7):

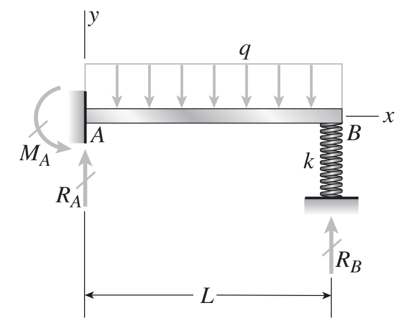
$$v = -\frac{\delta_B x^2}{2L^3}(3L - x) \quad \longleftarrow$$

SLOPE (FROM EQ. 6):

$$v' = -\frac{3\delta_B x}{2L^3}(2L - x)$$

Problem 10.3-4 A cantilever beam AB of length L has a fixed support at A and a spring support at B (see figure). The spring behaves in a linearly elastic manner with stiffness k.

If a uniform load of intensity q acts on the beam, what is the downward displacement δ_B of end B of the beam? (Use the second-order differential equation of the deflection curve, that is, the bending-moment equation.)



Solution 10.3-4 **Beam with spring support**

q = intensity of uniform load

EQUILIBRIUM $R_A = qL - R_B$ (1)

$$M_A = \frac{qL^2}{2} - R_B L \quad (2)$$

SPRING $R_B = k\delta_B$ (3)

δ_B = downward displacement of point B.

BENDING MOMENT (FROM EQUILIBRIUM)

$$M = R_A x - M_A - \frac{qx^2}{2}$$

DIFFERENTIAL EQUATIONS

$$EIv'' = M = R_A x - M_A - \frac{qx^2}{2}$$

$$EIv' = R_A\frac{x^2}{2} - M_A x - \frac{qx^3}{6} + C_1$$

$$EIv = R_A\frac{x^3}{6} - M_A\frac{x^2}{2} - \frac{qx^4}{24} + C_1 x + C_2$$

B.C. 1 $v'(0) = 0 \quad \therefore C_1 = 0$

B.C. 2 $v(0) = 0 \quad \therefore C_2 = 0$

B.C. 3 $v(L) = -\delta_B$

$$\therefore -EI\delta_B = \frac{R_A L^3}{6} - \frac{M_A L^2}{2} - \frac{qL^4}{24}$$

Substitute R_A and M_A from Eqs. (1) and (2):

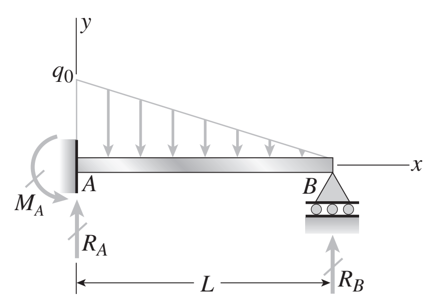
$$-EI\delta_B = \frac{R_B L^3}{3} - \frac{qL^4}{8}$$

Substitute for R_B from Eq. (3) and solve:

$$\delta_B = \frac{3qL^4}{24EI + 8kL^3} \quad \longleftarrow$$

Problem 10.3-5 A propped cantilever beam AB of length L supports a triangularly distributed load of maximum intensity q_0 (see figure).

Beginning with the fourth-order differential equation of the deflection curve (the load equation), obtain the reactions of the beam and the equation of the deflection curve.



Solution 10.3-5 **Propped cantilever beam**

Triangular load $q = q_0(L - x)/L$

DIFFERENTIAL EQUATIONS

$$EIv'''' = -q = -\frac{q_0}{L}(L - x) \quad (1)$$

$$EIv''' = V = -q_0 x + \frac{q_0 x^2}{2L} + C_1 \quad (2)$$

$$EIv'' = M = -\frac{q_0 x^2}{2} + \frac{q_0 x^3}{6L} + C_1 x + C_2 \quad (3)$$

$$EIv' = -\frac{q_0 x^3}{6} + \frac{q_0 x^4}{24L} + C_1\frac{x^2}{2} + C_2 x + C_3 \quad (4)$$

$$EIv = -\frac{q_0 x^4}{24} + \frac{q_0 x^5}{120L} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4 \quad (5)$$

B.C. 1 $v''(L) = 0$ $\quad \therefore C_1 L + C_2 = \dfrac{q_0 L^2}{3}$ (6)

B.C. 2 $v'(0) = 0$ $\quad \therefore C_3 = 0$

B.C. 3 $v(0) = 0$ $\quad \therefore C_4 = 0$

B.C. 4 $v(L) = 0$ $\quad \therefore C_1 L + 3C_2 = \dfrac{q_0 L^2}{5}$ (7)

Solve Eqs. (6) and (7):

$$C_1 = \frac{2q_0 L}{5} \qquad C_2 = -\frac{q_0 L^2}{15}$$

SHEAR FORCE (EQ. 2)

$$V = \frac{q_0}{10L}(4L^2 - 10Lx + 5x^2)$$

REACTIONS $\quad R_A = V(0) = \dfrac{2q_0 L}{5}$ ←

$R_B = -V(L) = \dfrac{q_0 L}{10}$ ←

From equilibrium:

$$M_A = \frac{q_0 L^2}{6} - R_B L = \frac{q_0 L^2}{15} \quad \leftarrow$$

DEFLECTION CURVE (FROM EQ. 5)

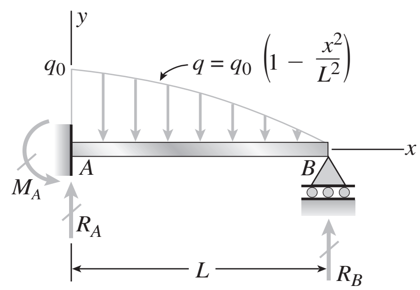
$$EIv = -\frac{q_0 x^4}{24} + \frac{q_0 x^5}{120L} + \frac{2q_0 L}{5}\left(\frac{x^3}{6}\right) - \frac{q_0 L^2}{15}\left(\frac{x^2}{2}\right)$$

or

$$v = -\frac{q_0 x^2}{120LEI}(4L^3 - 8L^2 x + 5Lx^2 - x^3) \quad \leftarrow$$

Problem 10.3-6 The load on a propped cantilever beam AB of length L is parabolically distributed according to the equation $q = q_0(1 - x^2/L^2)$, as shown in the figure.

Beginning with the fourth-order differential equation of the deflection curve (the load equation), obtain the reactions of the beam and the equation of the deflection curve.



Solution 10.3-6 Propped cantilever beam

Parabolic load $\quad q = q_0(1 - x^2/L^2)$

DIFFERENTIAL EQUATIONS

$$EIv'''' = -q = -q_0(1 - x^2/L^2) \quad (1)$$

$$EIv''' = V = -q_0(x - x^3/3L^2) + C_1 \quad (2)$$

$$EIv'' = M = -q_0\left(\frac{x^2}{2} - \frac{x^4}{12L^2}\right) + C_1 x + C_2 \quad (3)$$

$$EIv' = -q_0\left(\frac{x^3}{6} - \frac{x^5}{60L^2}\right) + C_1\frac{x^2}{2} + C_2 x + C_3 \quad (4)$$

$$EIv = -q_0\left(\frac{x^4}{24} - \frac{x^6}{360L^2}\right) + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4 \quad (5)$$

B.C. 1 $v''(L) = 0$ $\quad \therefore C_1 L + C_2 = 5q_0 L^2/12$ (6)

B.C. 2 $v'(0) = 0$ $\quad \therefore C_3 = 0$

B.C. 3 $v(0) = 0$ $\quad \therefore C_4 = 0$

B.C. 4 $v(L) = 0$ $\quad \therefore C_1 L + 3C_2 = 7q_0 L^2/30$ (7)

Solve Eqs. (6) and (7):

$$C_1 = 61q_0 L/120 \qquad C_2 = -11q_0 L^2/120$$

SHEAR FORCE (EQ. 2)

$$V = \frac{q_0}{120L^2}(61L^3 - 120L^2 x + 40x^3)$$

REACTIONS $\quad R_A = V(0) = 61q_0 L/120$ ←

$R_B = -V(L) = 19q_0 L/120$ ←

From equilibrium:

$$M_A = \frac{2}{3}(q_0)(L)\left(\frac{3L}{8}\right) - R_B L = \frac{11q_0 L^2}{120} \quad \leftarrow$$

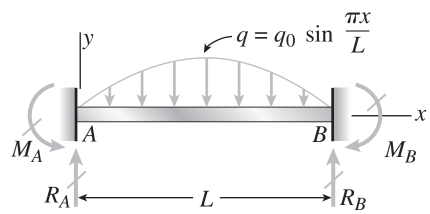
DEFLECTION CURVE (FROM EQ. 5)

$$v = -\frac{q_0 x^2}{720L^2 EI}(33L^4 - 61L^3 x + 30L^2 x^2 - 2x^4)$$

$$= -\frac{q_0 x^2(L - x)}{720L^2 EI}(33L^3 - 28L^2 x + 2Lx^2 + 2x^3) \quad \leftarrow$$

Problem 10.3-7 The load on a fixed-end beam AB of length L is distributed in the form of a sine curve (see figure). The intensity of the distributed load is given by the equation $q = q_0 \sin \pi x/L$.

Beginning with the fourth-order differential equation of the deflection curve (the load equation), obtain the reactions of the beam and the equation of the deflection curve.



Solution 10.3-7 **Fixed-end beam (sine load)**

$q = q_0 \sin \pi x/L$

FROM SYMMETRY: $R_A = R_B \qquad M_A = M_B$

DIFFERENTIAL EQUATIONS

$$EIv'''' = -q = -q_0 \sin \pi x/L \qquad (1)$$

$$EIv''' = V = \frac{q_0 L}{\pi} \cos \frac{\pi x}{L} + C_1 \qquad (2)$$

$$EIv'' = M = \frac{q_0 L^2}{\pi^2} \sin \frac{\pi x}{L} + C_1 x + C_2 \qquad (3)$$

$$EIv' = -\frac{q_0 L^3}{\pi^3} \cos \frac{\pi x}{L} + C_1 \frac{x^2}{2} + C_2 x + C_3 \qquad (4)$$

$$EIv = -\frac{q_0 L^4}{\pi^4} \sin \frac{\pi x}{L} + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4 \qquad (5)$$

B.C. 1 From symmetry, $V\left(\frac{L}{2}\right) = 0 \quad \therefore C_1 = 0$

B.C. 2 $v'(0) = 0 \quad \therefore C_3 = q_0 L^3/\pi^3$

B.C. 3 $v'(L) = 0 \quad \therefore C_2 = -2q_0 L^2/\pi^3$

B.C. 4 $v(0) = 0 \quad \therefore C_4 = 0$

SHEAR FORCE (EQ. 2)

$$V = \frac{q_0 L}{\pi} \cos \frac{\pi x}{L} \qquad R_A = V(0) = \frac{q_0 L}{\pi} \quad \longleftarrow$$

$$R_B = R_A = \frac{q_0 L}{\pi} \quad \longleftarrow$$

BENDING MOMENT (EQ. 3)

$$M = \frac{q_0 L^2}{\pi^3}\left(\pi \sin \frac{\pi x}{L} - 2\right)$$

$$M_A = -M(0) = \frac{2q_0 L^2}{\pi^3} \qquad M_B = M_A = \frac{2q_0 L^2}{\pi^3} \quad \longleftarrow$$

DEFLECTION CURVE (FROM EQ. 5)

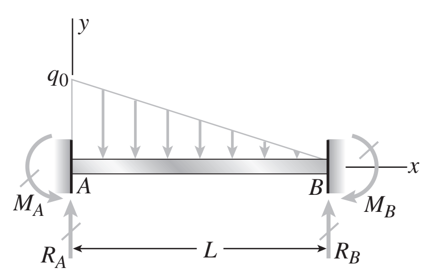
$$EIv = -\frac{q_0 L^4}{\pi^4} \sin \frac{\pi x}{L} - \frac{q_0 L^2 x^2}{\pi^3} + \frac{q_0 L^3 x}{\pi^3}$$

or

$$v = -\frac{q_0 L^2}{\pi^4 EI}\left(L^2 \sin \frac{\pi x}{L} + \pi x^2 - \pi L x\right) \quad \longleftarrow$$

Problem 10.3-8 A fixed-end beam AB of length L supports a triangularly distributed load of maximum intensity q_0 (see figure).

Beginning with the fourth-order differential equation of the deflection curve (the load equation), obtain the reactions of the beam and the equation of the deflection curve.



Solution 10.3-8 **Fixed-end beam (triangular load)**

$q = q_0(1 - x/L)$

DIFFERENTIAL EQUATIONS

$$EIv'''' = -q = -q_0\left(1 - \frac{x}{L}\right) \qquad (1)$$

$$EIv''' = V = -q_0\left(x - \frac{x^2}{2L}\right) + C_1 \qquad (2)$$

$$EIv'' = M = -q_0\left(\frac{x^2}{2} - \frac{x^3}{6L}\right) + C_1 x + C_2 \qquad (3)$$

$$EIv' = -q_0\left(\frac{x^3}{6} - \frac{x^4}{24L}\right) + C_1 \frac{x^2}{2} + C_2 x + C_3 \qquad (4)$$

$$EIv = -q_0\left(\frac{x^4}{24} - \frac{x^5}{120L}\right) + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4 \qquad (5)$$

B.C. 1 $v'(0) = 0$ $\therefore C_3 = 0$

B.C. 2 $v'(L) = 0$ $\therefore C_1L + 2C_2 = \dfrac{q_0L^2}{4}$ (6)

B.C. 3 $v(0) = 0$ $\therefore C_4 = 0$

B.C. 4 $v(L) = 0$ $\therefore C_1L + 3C_2 = \dfrac{q_0L^2}{5}$ (7)

Solve eqs. (6) and (7):

$$C_1 = \frac{7q_0L}{20} \quad C_2 = -\frac{q_0L^2}{20}$$

SHEAR FORCE (EQ. 2)

$$V = \frac{q_0}{20L}(7L^2 - 20Lx + 10x^2)$$

REACTIONS $R_A = V(0) = \dfrac{7q_0L}{20}$ ⟵

$R_B = -V(L) = \dfrac{3q_0L}{20}$ ⟵

BENDING MOMENT (EQ. 3)

$$M = -\frac{q_0}{60L}(3L^3 - 21L^2x + 30Lx^2 - 10x^3)$$

REACTIONS $M_A = -M(0) = \dfrac{q_0L^2}{20}$ ⟵

$M_B = -M(L) = \dfrac{q_0L^2}{30}$ ⟵

DEFLECTION CURVE (EQ. 5)

$$v = -\frac{q_0x^2}{120LEI}(3L^3 - 7L^2x + 5Lx^2 - x^3)$$

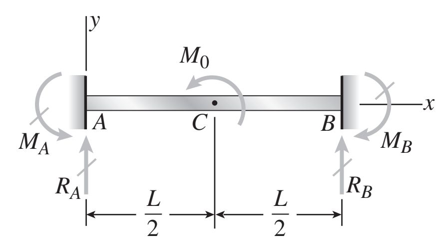
or

$$v = -\frac{q_0x^2}{120LEI}(L - x)^2(3L - x)$$ ⟵

Problem 10.3-9 A counterclockwise moment M_0 acts at the midpoint of a fixed-end beam ACB of length L (see figure).

Beginning with the second-order differential equation of the deflection curve (the bending-moment equation), determine all reactions of the beam and obtain the equation of the deflection curve for the left-hand half of the beam.

Then construct the shear-force and bending-moment diagrams for the entire beam, labeling all critical ordinates. Also, draw the deflection curve for the entire beam.



Solution 10.3-9 Fixed-end beam (M_0 = applied load)

Beam is symmetric; load is antisymmetric.

Therefore, $R_A = -R_B \quad M_A = -M_B \quad \delta_C = 0$

DIFFERENTIAL EQUATIONS ($0 \le x \le L/2$)

$$EIv'' = M = R_Ax - M_A \quad (1)$$

$$EIv' = R_A\frac{x^2}{2} - M_Ax + C_1 \quad (2)$$

$$EIv = R_A\frac{x^3}{6} - M_A\frac{x^2}{2} + C_1x + C_2 \quad (3)$$

B.C. 1 $v'(0) = 0$ $\therefore C_1 = 0$

B.C. 2 $v(0) = 0$ $\therefore C_2 = 0$

B.C. 3 $v\left(\dfrac{L}{2}\right) = 0$ $\therefore M_A = \dfrac{R_AL}{6}$ Also, $M_B = \dfrac{-R_AL}{6}$

EQUILIBRIUM (OF ENTIRE BEAM)

$\sum M_B = 0 \qquad M_A + M_0 - M_B - R_AL = 0$

or, $\dfrac{R_AL}{6} + M_0 + \dfrac{R_AL}{6} - R_AL = 0$

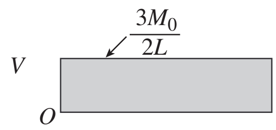
$\therefore R_A = -R_B = \dfrac{3M_0}{2L}$ ⟵

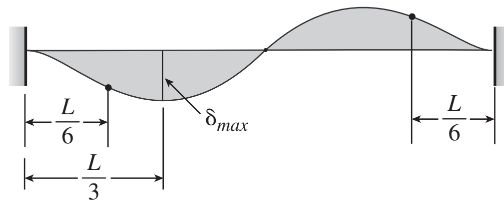
$M_A = \dfrac{R_AL}{6}$ $\therefore M_A = -M_B = \dfrac{M_0}{4}$ ⟵

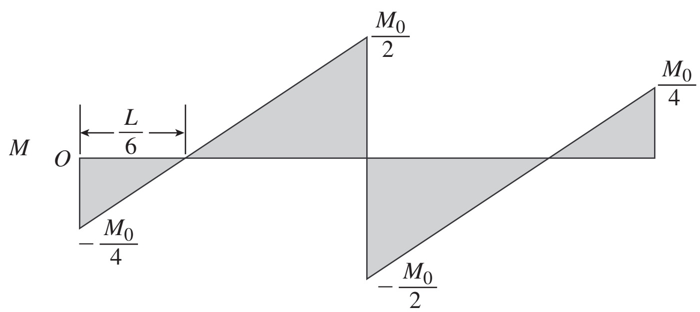
DEFLECTION CURVE (EQ. 3)

$$v = -\frac{M_0x^2}{8LEI}(L - 2x) \quad \left(0 \le x \le \frac{L}{2}\right)$$ ⟵

DIAGRAMS







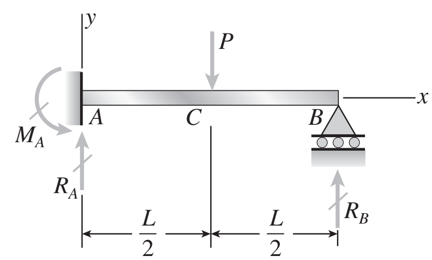
$$\delta_{max} = \frac{M_0 L^2}{216\,EI}$$

At point of inflection: $\delta = \delta_{max}/2$

Problem 10.3-10 A propped cantilever beam AB supports a concentrated load P acting at the midpoint C (see figure).

Beginning with the second-order differential equation of the deflection curve (the bending-moment equation), determine all reactions of the beam and draw the shear-force and bending-moment diagrams for the entire beam.

Also, obtain the equations of the deflection curves for both halves of the beam, and draw the deflection curve for the entire beam.



Solution 10.3-10 Propped cantilever beam

P = applied load at $x = L/2$

Select R_B as redundant reaction.

REACTIONS (FROM EQUILIBRIUM)

$$R_A = P - R_B \qquad (1) \qquad\qquad M_A = \frac{PL}{2} - R_B L \qquad (2)$$

BENDING MOMENTS (FROM EQUILIBRIUM)

$$M = R_A x - M_A = (P - R_B)x - \left(\frac{PL}{2} - R_B L\right) \qquad \left(0 \le x \le \frac{L}{2}\right)$$

$$M = R_B(L - x) \qquad \left(\frac{L}{2} \le x \le L\right)$$

DIFFERENTIAL EQUATIONS ($0 \le x \le L/2$)

$$EIv'' = M = (P - R_B)\,x - \left(\frac{PL}{2} - R_B L\right) \qquad (3)$$

$$EIv' = (P - R_B)\frac{x^2}{2} - \left(\frac{PL}{2} - R_B L\right)x + C_1 \qquad (4)$$

$$EIv = (P - R_B)\frac{x^3}{6} - \left(\frac{PL}{2} - R_B L\right)\frac{x^2}{2} + C_1 x + C_2 \qquad (5)$$

B.C. 1 $\quad v'(0) = 0 \qquad \therefore C_1 = 0$

B.C. 2 $\quad v(0) = 0 \qquad \therefore C_2 = 0$

DIFFERENTIAL EQUATIONS ($L/2 \le x \le L$)

$$EIv'' = M = R_B(L - x) \qquad (6)$$

$$EIv' = R_B L x - R_B\frac{x^2}{2} + C_3 \qquad (7)$$

$$EIv = R_B L\frac{x^2}{2} - R_B\frac{x^3}{6} + C_3 x + C_4 \qquad (8)$$

B.C. 3 $\quad v(L) = 0 \qquad \therefore C_3 L + C_4 = -\dfrac{R_B L^3}{3}$ (9)

B.C. 4 continuity condition at point C

$$\text{at } x = \frac{L}{2}: \quad (v')_{\text{Left}} = (v')_{\text{Right}}$$

$$(P - R_B)\left(\frac{L^2}{8}\right) - \left(\frac{PL}{2} - R_B L\right)\left(\frac{L}{2}\right) = R_B\left(\frac{L}{2}\right) - R_B\left(\frac{L^2}{8}\right) + C_3$$

$$\text{or } C_3 = -\frac{PL^2}{8} \qquad (10)$$

$$\text{From eq. (9): } C_4 = -\frac{R_B L^3}{3} + \frac{PL^3}{8} \qquad (11)$$

B.C. 5 continuity condition at point C.

$$\text{at } x = \frac{L}{2}: \quad (v)_{\text{Left}} = (v)_{\text{Right}}$$

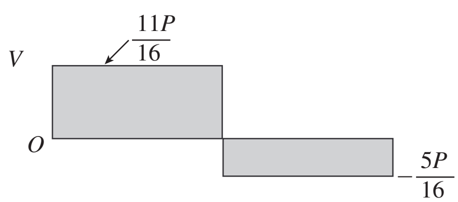
$$(P - R_B)\frac{L^3}{48} - \left(\frac{PL}{2} - R_B L\right)\frac{L^2}{8} = R_B L\left(\frac{L^2}{8}\right) - R_B\left(\frac{L^3}{48}\right) - \frac{PL^2}{8}\left(\frac{L}{2}\right) - \frac{R_B L^3}{3} + \frac{PL^3}{8}$$

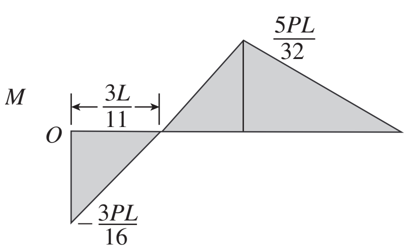
$$\text{or } R_B = \frac{5P}{16} \quad \longleftarrow$$

$$\text{From eq. (1): } \quad R_A = P - R_B = \frac{11P}{16} \quad \longleftarrow$$

$$\text{From eq. (2): } \quad M_A = \frac{PL}{2} - R_B L = \frac{3PL}{16} \quad \longleftarrow$$

SHEAR-FORCE AND BENDING MOMENT DIAGRAMS





DEFLECTION CURVE FOR $0 \le x \le L/2$ (FROM EQ. 5)

$$v = -\frac{Px^2}{96EI}(9L - 11x) \quad (0 \le x \le L/2) \quad \longleftarrow$$

DEFLECTION CURVE FOR $L/2 \le x \le L$ (FROM EQ. 8)

$$v = -\frac{P}{96EI}(-2L^3 + 12L^2x - 15Lx^2 + 5x^3)$$

$$= -\frac{P}{96EI}(L - x)(-2L^2 + 10Lx - 5x^2)$$

$$(L/2 \le x \le L) \quad \longleftarrow$$

SLOPE IN RIGHT-HAND PART OF THE BEAM

$$\text{From eq. (7): } v' = -\frac{P}{32EI}(4L^2 - 10Lx + 5x^2)$$

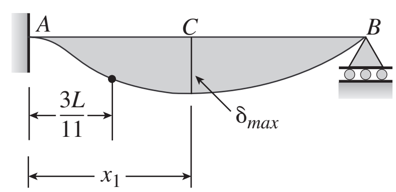
Point of zero slope:

$$5x_1^2 - 10Lx_1 + 4L^2 = 0 \quad x_1 = \frac{L}{5}\left(5 - \sqrt{5}\right) = 0.5528L$$

MAXIMUM DEFLECTION

$$\delta_{\max} = -(v)_{x=x_1} = 0.009317\frac{PL^3}{EI}$$

DEFLECTION CURVE

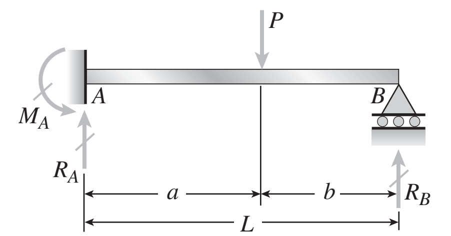


Method of Superposition

The problems for Section 10.4 are to be solved by the method of superposition. All beams have constant flexural rigidity EI unless otherwise stated. When drawing shear-force and bending-moment diagrams, be sure to label all critical ordinates, including maximum and minimum values.

Problem 10.4-1 A propped cantilever beam AB of length L carries a concentrated load P acting at the position shown in the figure.

Determine the reactions R_A, R_B, and M_A for this beam. Also, draw the shear-force and bending-moment diagrams, labeling all critical ordinates.



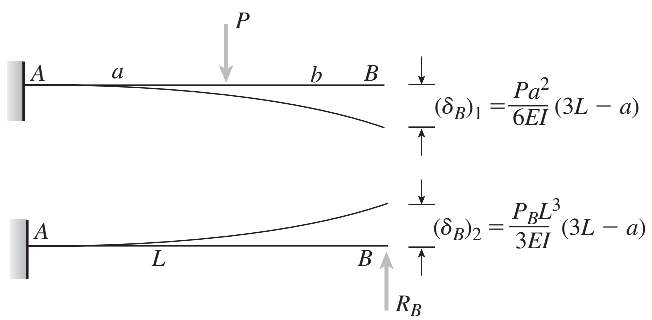
Solution 10.4-1 Propped cantilever beam

Select R_B as redundant.

EQUILIBRIUM

$R_A = P - R_B \qquad M_A = Pa - R_B L$

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



COMPATIBILITY

$$\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0$$

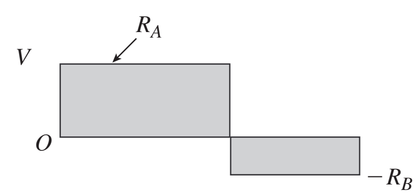
$$\delta_B = \frac{Pa^2}{6EI}(3L - a) - \frac{R_B L^3}{3EI} = 0$$

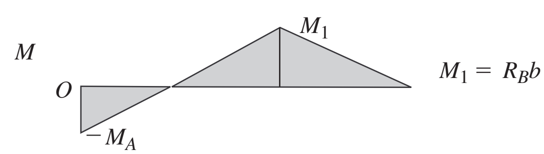
$$R_B = \frac{Pa^2}{2L^3}(3L - a) \quad \longleftarrow$$

OTHER REACTIONS (FROM EQUILIBRIUM)

$$R_A = \frac{Pb}{2L^3}(3L^2 - b^2) \quad M_A = \frac{Pab}{2L^2}(L + b) \quad \longleftarrow$$

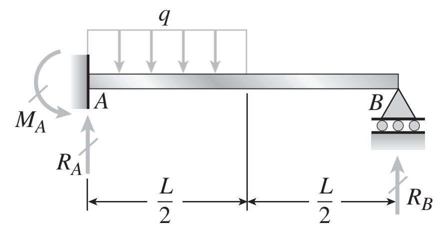
SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS





Problem 10.4-2 The propped cantilever beam shown in the figure supports a uniform load of intensity q on the left-hand half of the beam.

Find the reactions R_A, R_B, and M_A, and then draw the shear-force and bending-moment diagrams, labeling all critical ordinates.

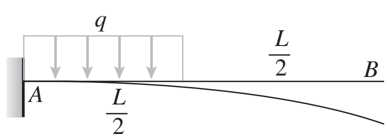


Solution 10.4-2 **Propped cantilever beam**

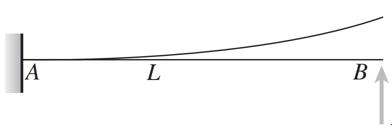
Select R_B as redundant.

EQUILIBRIUM $R_A = \frac{qL}{2} - R_B \quad M_A = \frac{qL^2}{8} - R_B L$

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



$$(\delta_B)_1 = \frac{PqL^4}{384EI}$$



$$(\delta_B)_2 = \frac{R_B L^3}{3EI}$$

COMPATIBILITY $\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0$

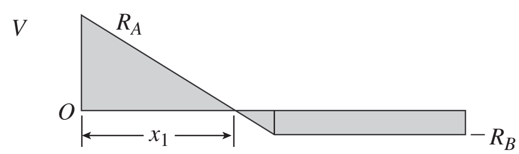
Substitute for $(\delta_B)_1$ and $(\delta_B)_2$ and solve for R_B:

$$R_B = \frac{7qL}{128} \quad \longleftarrow$$

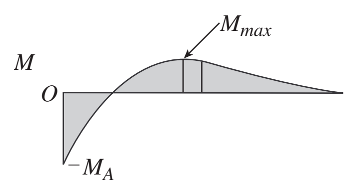
OTHER REACTIONS (FROM EQUILIBRIUM)

$$R_A = \frac{57qL}{128} \quad M_A = \frac{9qL^2}{128} \quad \longleftarrow$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS



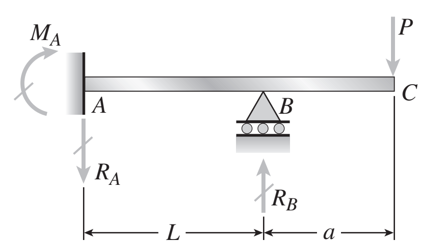
$$x_1 = \frac{57L}{128}$$



$$M_{\max} = \frac{945qL^2}{32{,}768}$$

Problem 10.4-3 The figure shows a propped cantilever beam ABC having span length L and an overhang of length a. A concentrated load P acts at the end of the overhang.

Determine the reactions R_A, R_B, and M_A for this beam. Also, draw the shear-force and bending-moment diagrams, labeling all critical ordinates.



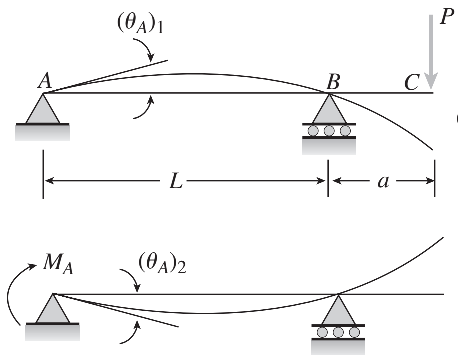
Solution 10.4-3 Beam with an overhang

Select M_A as redundant.

EQUILIBRIUM

$$R_A = \frac{1}{L}(M_A + Pa) \quad R_B = \frac{1}{L}(M_A + PL + Pa)$$

RELEASED STRUCTURE AND FORCE-DISPL. EQS.



$$(\theta_A)_1 = \frac{PAL}{6EI}$$

$$(\theta_A)_2 = \frac{M_A L}{3EI}$$

COMPATIBILITY $\theta_A = (\theta_A)_1 - (\theta_A)_2 = 0$

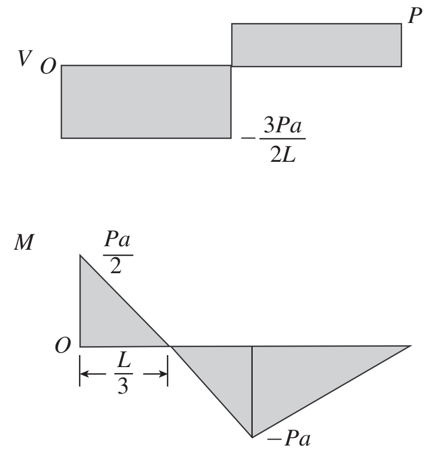
Substitute for $(\theta_A)_1$ and $(\theta_A)_2$ and solve for M_A:

$$M_A = \frac{Pa}{2} \quad \longleftarrow$$

OTHER REACTIONS (FROM EQUILIBRIUM)

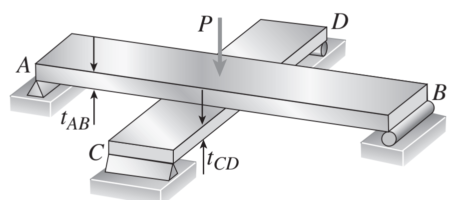
$$R_A = \frac{3Pa}{2L} \qquad R_B = \frac{P}{2L}(2L + 3a) \quad \longleftarrow$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS



Problem 10.4-4 Two flat beams *AB* and *CD*, lying in horizontal planes, cross at right angles and jointly support a vertical load *P* at their midpoints (see figure). Before the load *P* is applied, the beams just touch each other. Both beams are made of the same material and have the same widths. Also, the ends of both beams are simply supported. The lengths of beams *AB* and *CD* are L_{AB} and L_{CD}, respectively.

What should be the ratio t_{AB}/t_{CD} of the thicknesses of the beams if all four reactions are to be the same?



Solution 10.4-4 Two beams supporting a load P

For all four reactions to be the same, each beam must support one-half of the load *P*.

DEFLECTIONS

$$\delta_{AB} = \frac{(P/2)L_{AB}^3}{48EI_{AB}} \qquad \delta_{CD} = \frac{(P/2)L_{CD}^3}{48EI_{CD}}$$

COMPATIBILITY

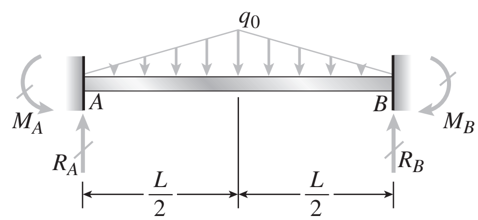
$$\delta_{AB} = \delta_{CD} \quad \text{or} \quad \frac{L_{AB}^3}{I_{AB}} = \frac{L_{CD}^3}{I_{CD}}$$

MOMENT OF INERTIA

$$I_{AB} = \frac{1}{12}bt_{AB}^3 \qquad I_{CD} = \frac{1}{12}bt_{CD}^3$$

$$\therefore \frac{L_{AB}^3}{t_{AB}^3} = \frac{L_{CD}^3}{t_{CD}^3} \qquad \frac{t_{AB}}{t_{CD}} = \frac{L_{AB}}{L_{CD}} \quad \longleftarrow$$

Problem 10.4-5 Determine the fixed-end moments (M_A and M_B) and fixed-end forces (R_A and R_B) for a beam of length L supporting a triangular load of maximum intensity q_0 (see figure). Then draw the shear-force and bending-moment diagrams, labeling all critical ordinates.

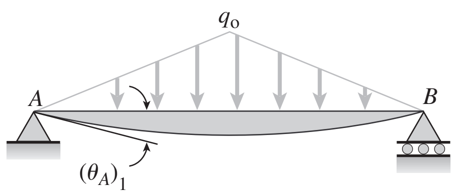


Solution 10.4-5 Fixed-end beam (triangular load)

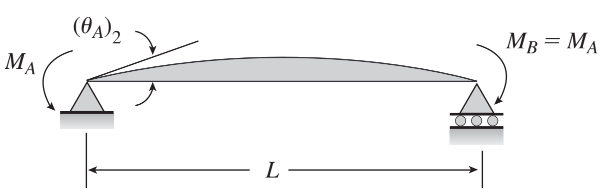
Select M_A and M_B as reduntants.

SYMMETRY $M_A = M_B$ $R_A = R_B$

EQUILIBRIUM $R_A = R_B = q_0L/4$ ←



$$(\theta_A)_1 = \frac{5q_0L^2}{192EI}$$



$$(\theta_A)_2 = \frac{M_AL}{2EI}$$

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS

COMPATIBILITY $\theta_A = (\theta_A)_1 - (\theta_A)_2 = 0$

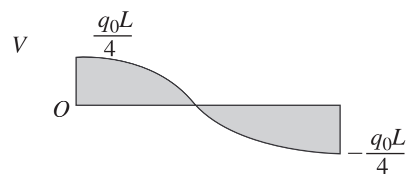
Substitute for $(\theta_A)_1$ and $(\theta_A)_2$ and solve for M_A:

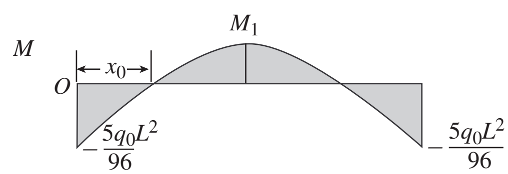
$$M_A = M_B = \frac{5q_0L^2}{96} \quad \leftarrow$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS

$$M_1 = \frac{q_0L^2}{32}$$

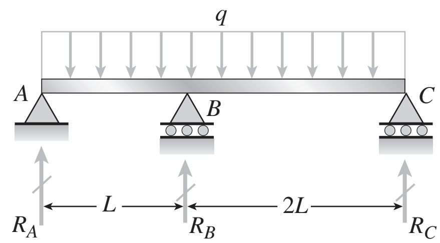
$$x_0 = 0.2231L$$





Problem 10.4-6 A continuous beam ABC with two unequal spans, one of length L and one of length $2L$, supports a uniform load of intensity q (see figure).

Determine the reactions R_A, R_B, and R_C for this beam. Also, draw the shear-force and bending-moment diagrams, labeling all critical ordinates.



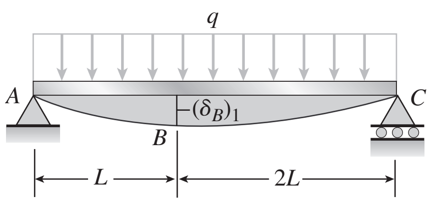
Solution 10.4-6 Continuous beam with two spans

Select R_B as redundant.

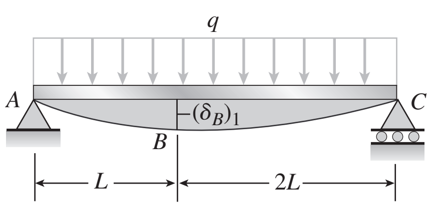
EQUILIBRIUM

$$R_A = \frac{3qL}{2} - \frac{2}{3}R_B \qquad R_C = \frac{3qL}{2} - \frac{1}{3}R_B$$

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



$$(\delta_B)_1 = \frac{11qL^4}{12\,EI}$$



$$(\delta_B)_2 = \frac{4R_BL^3}{9\,EI}$$

COMPATIBILITY

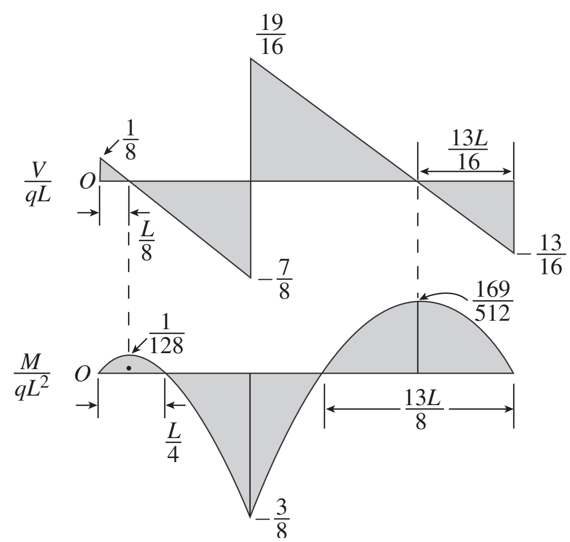
$$\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0$$

$$\frac{11qL^4}{12EI} - \frac{4R_BL^3}{9EI} = 0 \qquad R_B = \frac{33qL}{16} \quad \longleftarrow$$

OTHER REACTIONS (FROM EQUILIBRIUM)

$$R_A = \frac{qL}{8} \qquad R_C = \frac{13qL}{16} \quad \longleftarrow$$

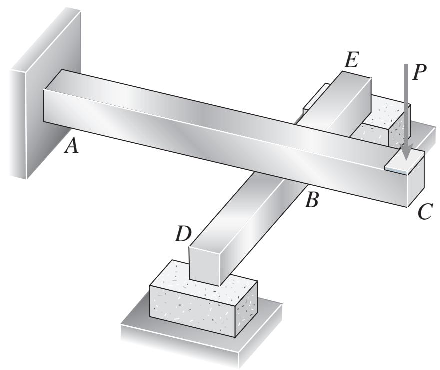
SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS

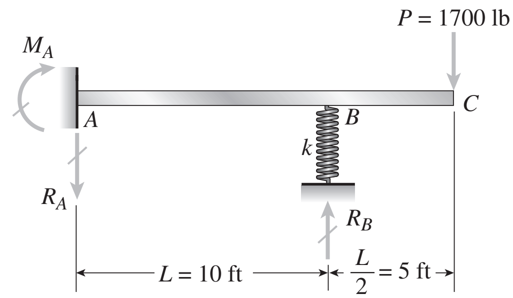


Problem 10.4-7 Beam ABC is fixed at support A and rests (at point B) upon the midpoint of beam DE (see the first part of the figure). Thus, beam ABC may be represented as a propped cantilever beam with an overhang BC and a linearly elastic support of stiffness k at point B (see the second part of the figure).

The distance from A to B is $L = 10$ ft, the distance from B to C is $L/2 = 5$ ft, and the length of beam DE is $L = 10$ ft. Both beams have the same flexural rigidity EI. A concentrated load $P = 1700$ lb acts at the free end of beam ABC.

Determine the reactions R_A, R_B, and M_A for beam ABC. Also, draw the shear-force and bending-moment diagrams for beam ABC, labeling all critical ordinates.





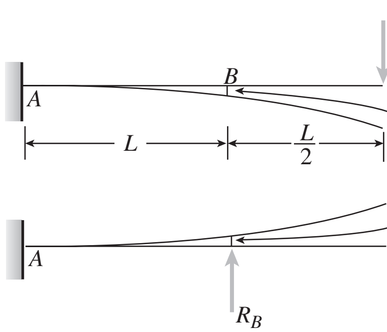
Solution 10.4-7 Beam with spring support

Select R_B as redundant.

EQUILIBRIUM

$R_A = R_B - P \qquad M_A = R_B L - 3PL/2$

RELEASED STRUCTURE AND FORCE-DISPL. EQS.



$(\delta_B)_1 = \dfrac{7PL^3}{12EI}$

$(\delta_B)_2 = \dfrac{R_B L^3}{3EI}$

COMPATIBILITY $\delta_B = (\delta_B)_1 - (\delta_B)_2 = \dfrac{R_B}{k}$

Beam DE: $k = \dfrac{48\,EI}{L^3}$

$$\frac{7PL^3}{12\,EI} - \frac{R_B L^3}{3\,EI} = \frac{R_B L^3}{48 EI} \qquad R_B = \frac{28P}{17} \quad \longleftarrow$$

OTHER REACTIONS (FROM EQUILIBRIUM)

$$R_A = \frac{11P}{17} \qquad M_A = \frac{5PL}{34} \quad \longleftarrow$$

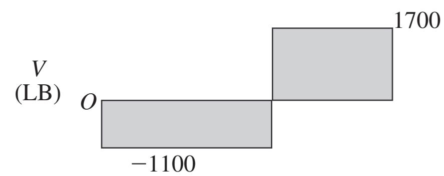
NUMERICAL VALUES

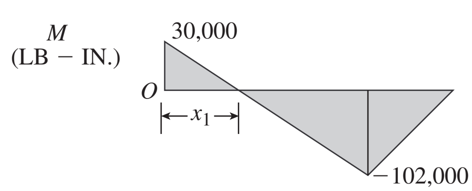
$P = 1700$ lb $\quad L = 10$ ft $= 120$ in.

$R_A = 1100$ lb $\quad R_B = 2800$ lb
$M_A = 30{,}000$ lb-in. $\quad \longleftarrow$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS

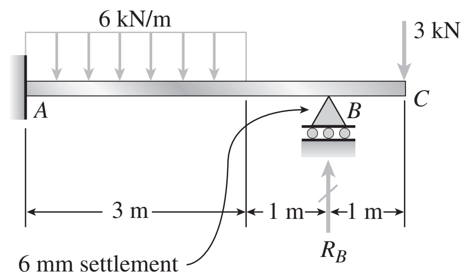
$x_1 = \dfrac{300}{11}$ in.
$= 27.27$ in.





Problem 10.4-8 The beam *ABC* shown in the figure has flexural rigidity $EI = 4.0$ MN·m^2. When the loads are applied to the beam, the support at *B* settles vertically downward through a distance of 6.0 mm.

Calculate the reaction R_B at support *B*.

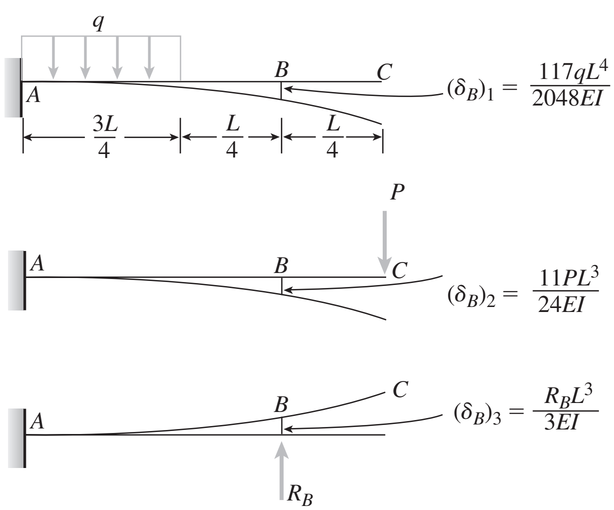


Solution 10.4-8 Overhanging beam with support settlement

Select R_B as redundant.

Δ = settlement of support *B*

RELEASED STRUCTURE AND FORCE-DISPL. EQS.



COMPATIBILITY $\delta_B = (\delta_B)_1 + (\delta_B)_2 - (\delta_B)_3 = \Delta$

Substitute for $(\delta_B)_1$, $(\delta_B)_2$, and $(\delta_B)_3$ and solve for R_B:

$$R_B = \frac{1}{2048}\left(351qL + 2816P - 6144\frac{EI\Delta}{L^3}\right) \quad \longleftarrow$$

NUMERICAL VALUES

$q = 6.0$ kN/m $P = 3.0$ kN $\Delta = 6.0$ mm
$L = 4.0$ m $EI = 4.0$ MN · m^2

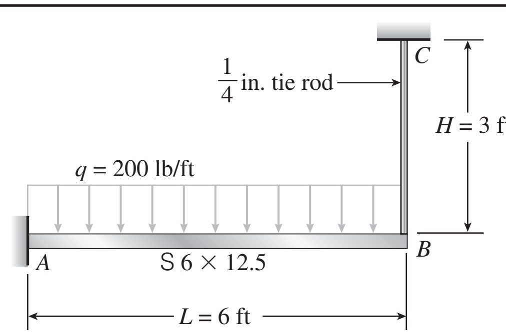
SUBSTITUTE INTO THE EQUATION FOR R_B

$R_B = 7.11$ kN $\longleftarrow$

Problem 10.4-9 A beam *AB* is cantilevered from a wall at one end and held by a tie rod at the other end (see figure). The beam is an S 6 × 12.5 I-beam section with length $L = 6$ ft. The tie rod has a diameter of 1/4 inch and length $H = 3$ ft. Both members are made of steel with $E = 30 \times 10^6$ psi. A uniform load of intensity $q = 200$ lb/ft acts along the length of the beam. Before the load *q* is applied, the tie rod just meets the end of the cable.

(a) Determine the tensile force *T* in the tie rod due to the uniform load *q*.

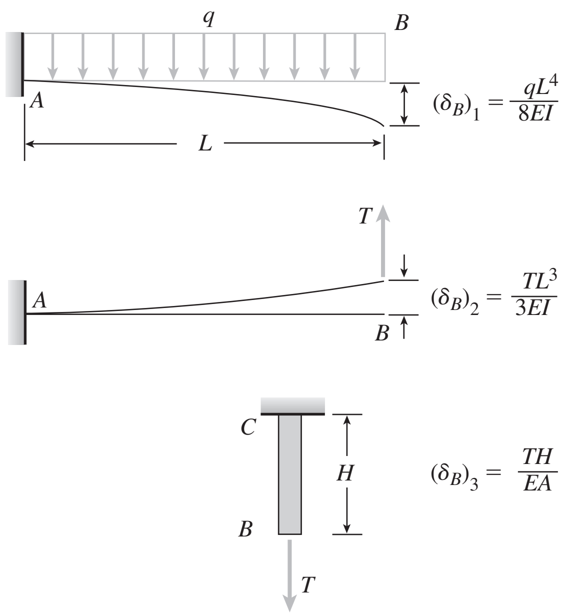
(b) Draw the shear-force and bending-moment diagrams for the beam, labeling all critical ordinates.



Solution 10.4-9 Beam supported by a tie rod

Select the force T in the tie rod as redundant.

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



COMPATIBILITY $(\delta_B)_1 - (\delta_B)_2 = (\delta_B)_3$

or $\frac{qL^4}{8EI} - \frac{TL^3}{3EI} = \frac{TH}{EA}$

$$T = \frac{3qAL^4}{8AL^3 + 24IH} \longleftarrow$$

NUMERICAL VALUES

$q = 200$ lb/ft $\quad L = 6$ ft $\quad H = 3$ ft

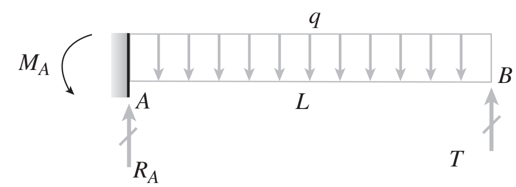
$E = 30 \times 10^6$ psi

Beam: S $6 \times 12.5 \quad I = 22.1$ in.^{4}

Tie Rod: $d = 0.25$ in. $\quad A = 0.04909$ in.^{2}

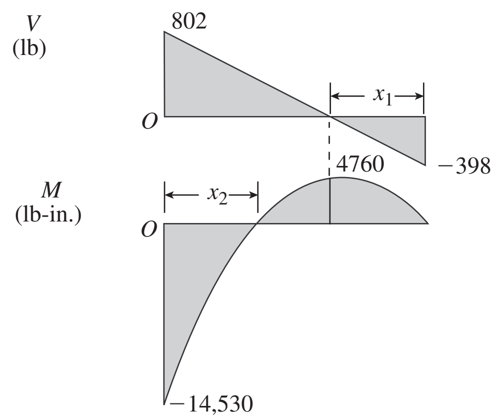
Substitute: $T = 398$ lb $\longleftarrow$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS



$R_A = qL - T = 802$ lb

$M_A = \frac{qL^2}{2} - TL = 14{,}530$ lb-in.

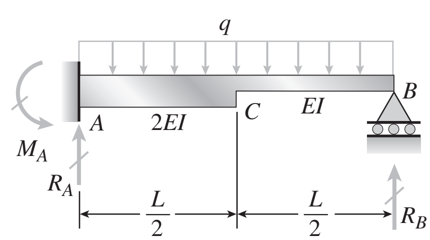


$x_1 = 23.9$ in.

$x_2 = 24.2$ in.

Problem 10.4-10 The figure shows a nonprismatic, propped cantilever beam AB with flexural rigidity $2EI$ from A to C and EI from C to B.

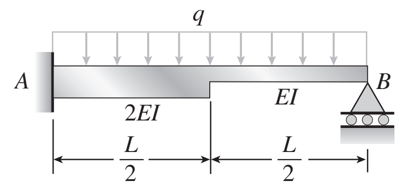
Determine all reactions of the beam due to the uniform load of intensity q. (*Hint:* Use the results of Problems 9.7-1 and 9.7-2.)



Solution 10.4-10 Nonprismatic beam

Select R_B as redundant.

RELEASED STRUCTURE



$(\delta_B)_1$ = downward deflection of end B due to load q

$(\delta_B)_2$ = upward deflection due to reaction R_B

FORCE-DISPLACEMENT RELATIONS

From prob. 9.7-2: $\delta_B = \dfrac{qL^4}{128EI_1}\left(1 + 15\dfrac{I_1}{I_2}\right)$

$I_1 \rightarrow I \quad I_2 \rightarrow 2I \qquad \therefore (\delta_B)_1 = \dfrac{17\,qL^4}{256\,EI}$

From prob. 9.7-1:

$$\delta_B = \frac{PL^3}{24\,EI_1}\left(1 + 7\frac{I_1}{I_2}\right) \qquad \therefore (\delta_B)_2 = \frac{3R_BL^3}{16\,EI}$$

COMPATIBILITY

$\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0$

or

$$\frac{17qL^4}{256EI} - \frac{3R_BL^3}{16EI} = 0 \quad R_B = \frac{17qL}{48} \quad \longleftarrow$$

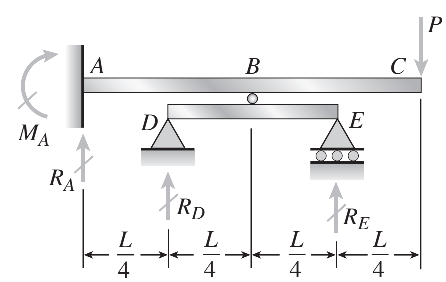
EQUILIBRIUM

$$R_A = qL - R_B = \frac{31qL}{48} \quad M_A = \frac{qL^2}{2} - R_BL = \frac{7qL^2}{48} \quad \longleftarrow$$

Problem 10.4-11 A beam ABC is fixed at end A and supported by beam DE at point B (see figure). Both beams have the same cross section and are made of the same material.

(a) Determine all reactions due to the load P.

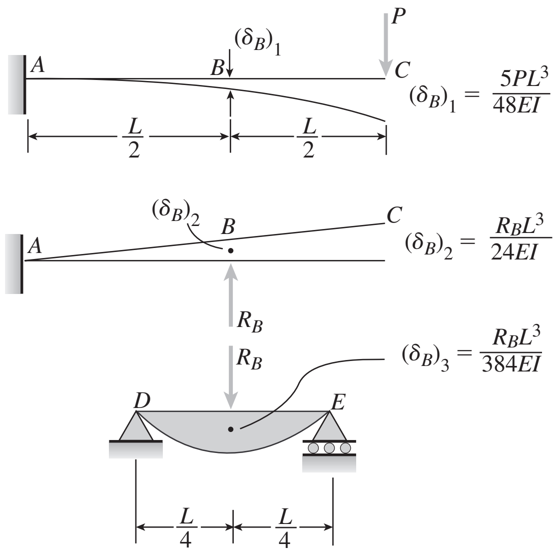
(b) What is the numerically largest bending moment in either beam?



Solution 10.4-11 Beam supported by a beam

Let R_B = interaction force between beams select R_B as redundant.

RELEASED STRUCTURE AND FORCE-DISPL. EQS.



COMPATIBILITY $(\delta_B)_1 - (\delta_B)_2 = (\delta_B)_3$

Substitute and solve: $R_B = \frac{40P}{17}$ ←

SYMMETRY AND EQUILIBRIUM

$$R_D = R_E = \frac{R_B}{2} = \frac{20P}{17} \quad \leftarrow$$

$$R_A = P - R_D - R_E = -\frac{23P}{17} \quad \leftarrow$$

(minus means downward)

$$M_A = R_B\left(\frac{L}{2}\right) - PL = \frac{3PL}{17} \quad \leftarrow$$

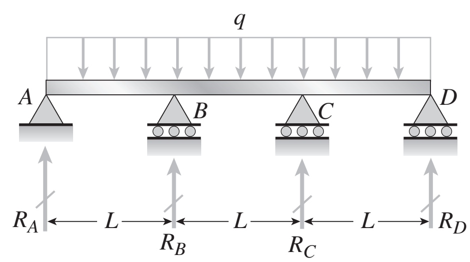
BEAM *ABC*: $M_{max} = M_B = -\frac{PL}{2}$

BEAM *DE*: $M_{max} = M_B = \frac{5PL}{17}$

$$|M_{max}| = \frac{PL}{2} \quad \leftarrow$$

Problem 10.4-12 A three-span continuous beam *ABCD* with three equal spans supports a uniform load of intensity *q* (see figure).

Determine all reactions of this beam and draw the shear-force and bending-moment diagrams, labeling all critical ordinates.



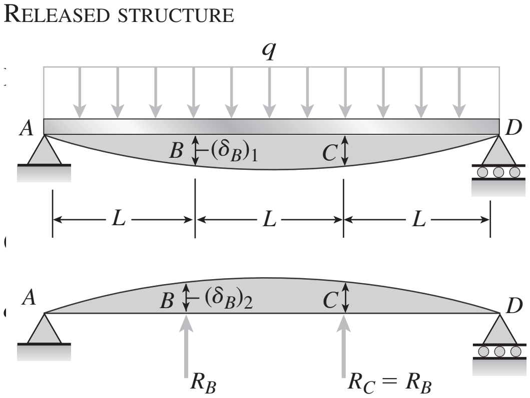
Solution 10.4-12 Three-span continuous beam

SELECT R_B AND R_C AS REDUNDANTS.

SYMMETRY AND EQUILIBRIUM

$$R_C = R_B \quad R_A = R_D = \frac{3qL}{2} - R_B$$

RELEASED STRUCTURE



FORCE-DISPLACEMENT RELATIONS

$$(\delta_B)_1 = \frac{11qL^4}{12EI} \quad (\delta_B)_2 = \frac{5R_BL^3}{6EI}$$

COMPATIBILITY

$$\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0 \quad \therefore R_B = \frac{11qL}{10} \quad \leftarrow$$

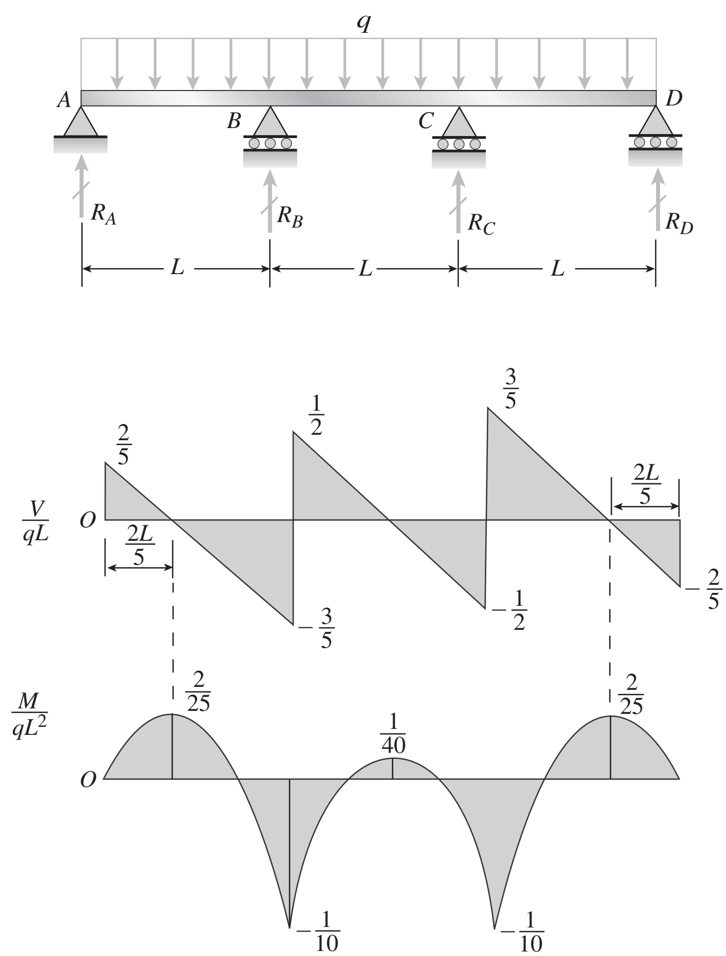
OTHER REACTIONS

From symmetry and equilibrium:

$$R_C = R_B = \frac{11qL}{10} \quad \leftarrow$$

$$R_A = R_D = \frac{2qL}{5} \quad \leftarrow$$

LOADING, SHEAR-FORCE, AND BENDING-MOMENT DIAGRAMS



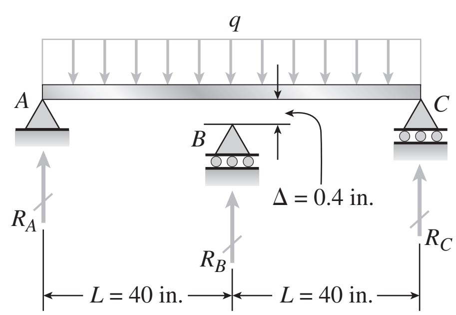
$$M_B = M_C = -\frac{qL^2}{10}$$

$$M_{\max} = \frac{2\,qL^2}{25}$$

Problem 10.4-13 A beam AC rests on simple supports at points A and C (see figure). A small gap $\Delta = 0.4$ in. exists between the unloaded beam and a support at point B, which is midway between the ends of the beam. The beam has total length $2L = 80$ in. and flexural rigidity $EI = 0.4 \times 10^9$ lb-in.^{2}

Plot a graph of the bending moment M_B at the midpoint of the beam as a function of the intensity q of the uniform load.

Hints: Begin by determining the intensity q_0 of the load that will just close the gap. Then determine the corresponding bending moment $(M_B)_0$. Next, determine the bending moment M_B (in terms of q) for the case where $q < q_0$. Finally, make a statically indeterminate analysis and determine the moment M_B (in terms of q) for the case where $q > q_0$. Plot M_B (units of lb-in.) versus q (units of lb/in.) with q varying from 0 to 2500 lb/in.



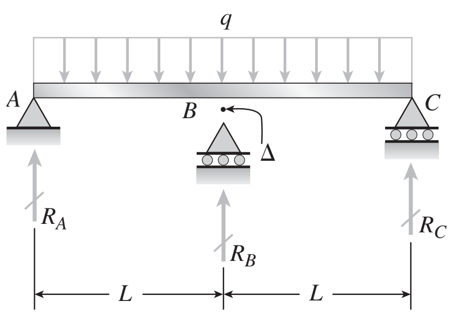
Solution 10.4-13 Beam on a support with a gap

q_0 = load required to close the gap

Δ = magnitude of gap

$(M_B)_0$ = bending moment when $q = q_0$

CASE 1 $\quad q < q_0$



$$\delta_B = \frac{5qL^4}{24EI}$$

$$M_B = \frac{qL^2}{2}$$

$$R_A = R_C = qL$$

CASE 2 $\quad q = q_0$

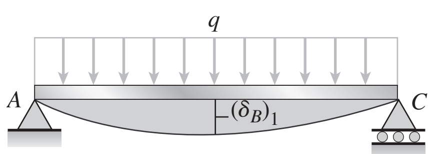
$$\delta_B = \Delta = \frac{5q_0L^4}{24EI} \quad q_0 = \frac{24EI\Delta}{5L^4} \qquad (1)$$

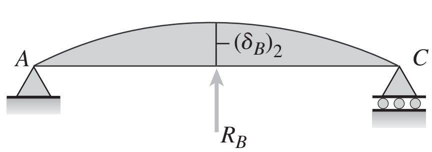
$$(M_B)_0 = \frac{q_0L^2}{2} = \frac{12EI\Delta}{5L^2} \qquad (2)$$

CASE 3 $\quad q > q_0$ (statically indeterminate)

Select R_B as redundant.

RELEASED STRUCTURE





$$(\delta_B)_1 = \frac{5qL^4}{24EI}$$

$$(\delta_B)_2 = \frac{R_BL^3}{6EI}$$

COMPATIBILITY $\quad \delta_B = (\delta_B)_1 - (\delta_B)_2 = \Delta$

$$\text{or} \quad \frac{5qL^4}{24EI} - \frac{R_BL^3}{6EI} = \Delta \quad R_B = \frac{5qL}{4} - \frac{6EI\Delta}{L^3}$$

EQUILIBRIUM

$$R_A = R_C \quad 2R_A - 2qL + R_B = 0$$

$$R_A = R_C = \frac{3qL}{8} + \frac{3EI\Delta}{L^3}$$

$$M_B = R_AL - \frac{qL^2}{2} = \frac{3EI\Delta}{L^2} - \frac{qL^2}{8}$$

NUMERICAL VALUES

$\Delta = 0.4$ in. $\quad L = 40$ in. $\quad EI = 0.4 \times 10^9$ lb-in.^{2}

Units: lb, in.

From eqs. (1) and (2): $\quad q_0 = 300$ lb/in.

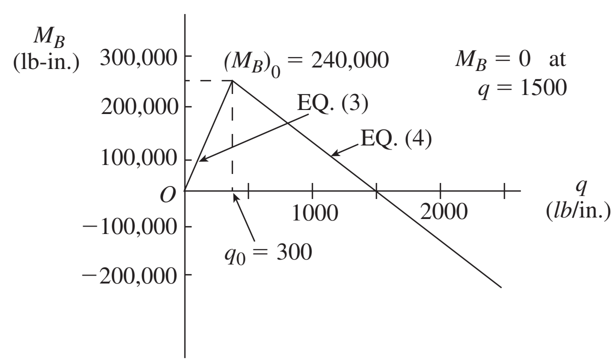
$(M_B)_0 = 240{,}000$ lb-in.

For $q < q_0$: $M_B = 800\,q \qquad (3)$

For $q > q_0$: $M_B = 300{,}000 - 200\,q \qquad (4)$

GRAPH OF BENDING MOMENT M_B (EQS. 3 AND 4)

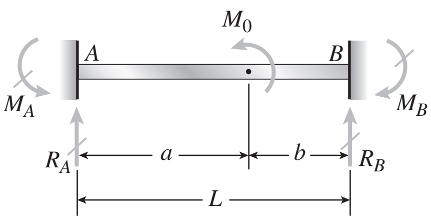
$M_B = 0$ at $q = 1500$ ⟵



Problem 10.4-14 A fixed-end beam AB of length L is subjected to a moment M_0 acting at the position shown in the figure.

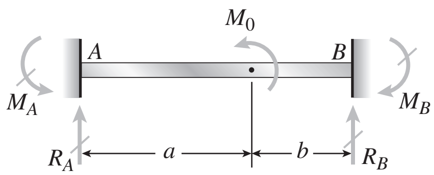
(a) Determine all reactions for this beam.

(b) Draw shear-force and bending-moment diagrams for the special case in which $a = b = L/2$.



Solution 10.4-14 **Fixed-end beam (M_0 = applied load)**

Select R_B and M_B as redundants.

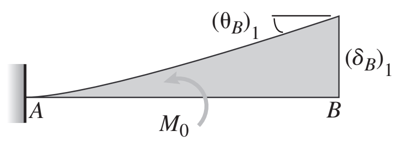


$L = a + b$

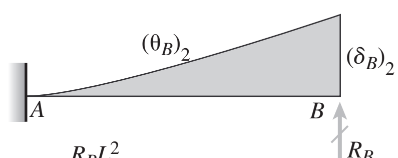
EQUILIBRIUM

$R_A = -R_B \qquad M_A = M_B - R_B L - M_0$

RELEASED STRUCTURE AND FORCE-DISPL. EQS.

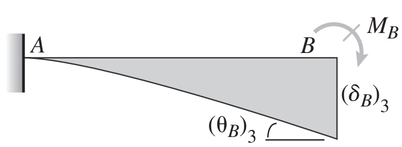


$(\theta_B)_1 = \dfrac{M_0 a}{EI} \qquad (\delta_B)_1 = \dfrac{M_0 a}{2EI}(a + 2b)$



$(\theta_B)_2 = \dfrac{R_B L^2}{2EI}$

$(\delta_B)_2 = \dfrac{R_B L^3}{3EI}$



$(\theta_B)_3 = \dfrac{M_B L}{EI} \qquad (\delta_B)_3 = \dfrac{M_B L^2}{2EI}$

$$(\theta_B)_1 = \frac{M_0 a}{EI} \qquad (\delta_B)_1 = \frac{M_0 a}{2EI}(a + 2b)$$

$$(\theta_B)_2 = \frac{R_B L^2}{2EI} \qquad (\delta_B)_2 = \frac{R_B L^3}{3EI}$$

$$(\theta_B)_3 = \frac{M_B L}{EI} \qquad (\delta_B)_3 = \frac{M_B L^2}{2EI}$$

COMPATIBILITY

$\delta_B = -(\delta_B)_1 - (\delta_B)_2 + (\delta_B)_3 = 0$

or $\quad 2R_B L^3 - 3M_B L^2 = -3M_0 a(a + 2b) \qquad (1)$

$\theta_B = (\theta_B)_1 + (\theta_B)_2 - (\theta_B)_3 = 0$

or $\quad R_B L^2 - 2M_B L = -2M_0 a \qquad (2)$

SOLVE EQS. (1) AND (2):

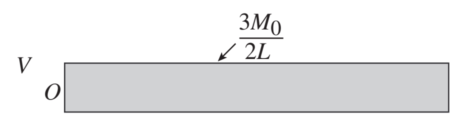
$$R_B = -\frac{6M_0 ab}{L^3} \qquad M_B = -\frac{M_0 a}{L^2}(3b - L) \quad \longleftarrow$$

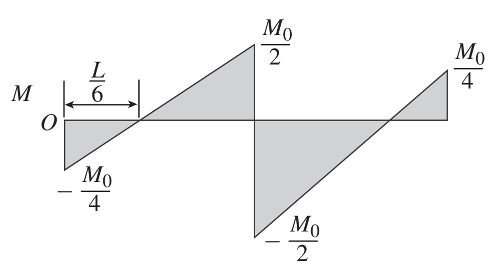
FROM EQUILIBRIUM:

$$R_A = \frac{6M_0 ab}{L^3} \qquad M_A = \frac{M_0 b}{L^2}(3a - L) \quad \longleftarrow$$

SPECIAL CASE $a = b = L/2$

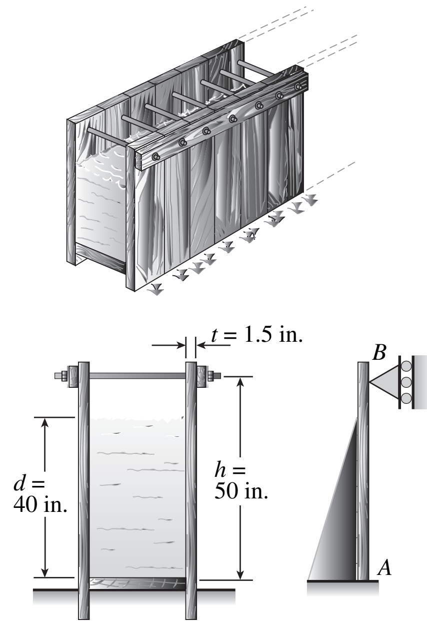
$$R_A = -R_B = \frac{3M_0}{2L} \qquad M_A = -M_B = \frac{M_0}{4}$$



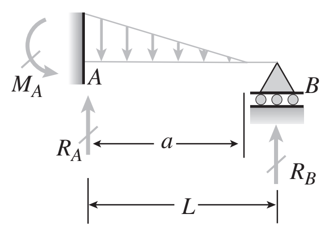


Problem 10.4-15 A temporary wood flume serving as a channel for irrigation water is shown in the figure. The vertical boards forming the sides of the flume are sunk in the ground, which provides a fixed support. The top of the flume is held by tie rods that are tightened so that there is no deflection of the boards at that point. Thus, the vertical boards may be modeled as a beam AB, supported and loaded as shown in the last part of the figure.

Assuming that the thickness t of the boards is 1.5 in., the depth d of the water is 40 in., and the height h to the tie rods is 50 in., what is the maximum bending stress σ in the boards? (*Hint:* The numerically largest bending moment occurs at the fixed support.)



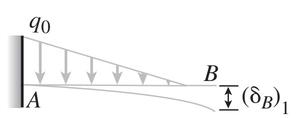
Solution 10.4-15 **Side wall of a wood flume**



Select R_B as redundant.

Equilibrium: $M_A = \dfrac{q_0 a^2}{6} - R_B L$

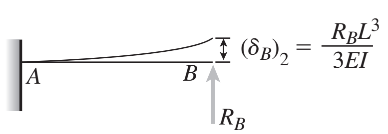
RELEASED STRUCTURE AND FORCE-DISPL. EQS.



From Table G-1, Case B:

$$(\delta_B)_1 = \frac{q_0 a^4}{30EI} + \frac{q_0 a^3}{24\,EI}(L - a) = \frac{q_0 a^3}{120\,EI}(5L - a)$$

$$(\delta_B)_2 = \frac{R_B L^3}{3\,EI}$$



COMPATIBILITY

$$\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0 \quad \therefore R_B = \frac{q_0 a^3(5L - a)}{40\,L^3}$$

MAXIMUM BENDING MOMENT

$$M_{\max} = M_A = \frac{1}{6} q_0 a^2 - R_B L$$

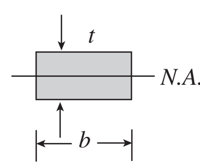
$$= \frac{q_0 a^2}{120\,L^2}(20\,L^2 - 15\,aL + 3a^2)$$

NUMERICAL VALUES

$a = 40$ in. $\quad L = 50$ in. $\quad t = 1.5$ in.
b = width of beam

$$S = \frac{bt^2}{6} \quad \sigma = \frac{M_{\max}}{S}$$

$\gamma = 62.4$ lb/ft^3 $= 0.03611$ lb/in.^{3}



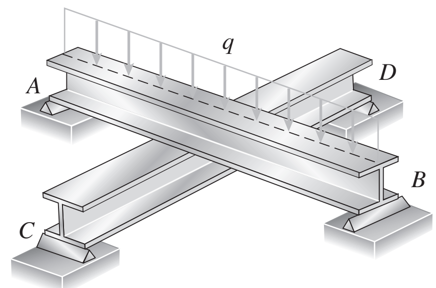
Pressure $p = \gamma a \quad q_0 = pb = \gamma ab$

$$M_{\max} = \frac{\gamma a^3 b}{120\,L^2}(20\,L^2 - 15\,aL + 3a^2) = 19605\,b$$

$$S = \frac{bt^2}{6} = 0.3750\,b \quad \sigma = \frac{M_{\max}}{S} = 509 \text{ psi} \quad \longleftarrow$$

Problem 10.4-16 Two identical, simply supported beams AB and CD are placed so that they cross each other at their midpoints (see figure). Before the uniform load is applied, the beams just touch each other at the crossing point.

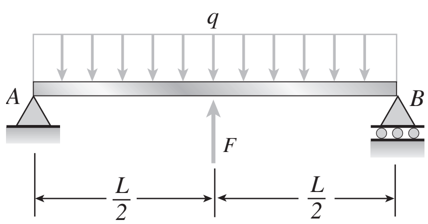
Determine the maximum bending moments $(M_{AB})_{max}$ and $(M_{CD})_{max}$ in beams AB and CD, respectively, due to the uniform load if the intensity of the load is $q = 6.4$ kN/m and the length of each beam is $L = 4$ m.



Solution 10.4-16 Two beams that cross

F = interaction force between the beams

UPPER BEAM



$(\delta_B)_1$ = downward deflection due to q

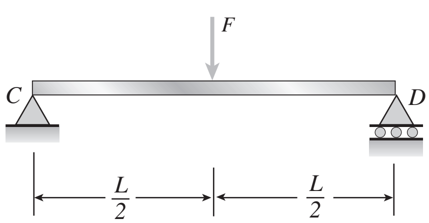
$= \dfrac{5qL^4}{384EI}$

$(\delta_B)_2$ = upward deflection due to F

$= \dfrac{FL^3}{48EI}$

$$\delta_{AB} = (\delta_B)_1 - (\delta_B)_2 = \frac{5qL^4}{384EI} - \frac{FL^3}{48EI}$$

LOWER BEAM

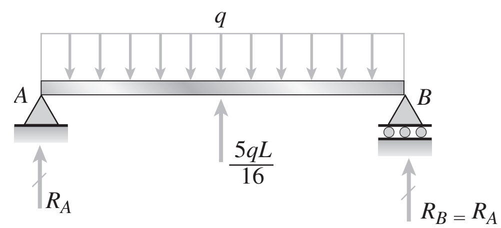


$$\delta_{CD} = \frac{FL^3}{48EI}$$

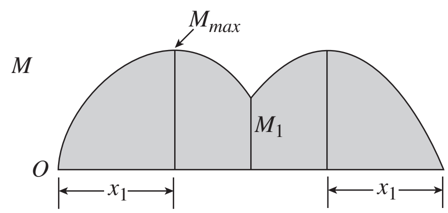
COMPATIBILITY $\delta_{AB} = \delta_{CD}$

$$\frac{5qL^4}{384EI} - \frac{FL^3}{48EI} = \frac{FL^3}{48EI} \quad \therefore F = \frac{5qL}{16}$$

UPPER BEAM



$$R_A = \frac{11qL}{32}$$



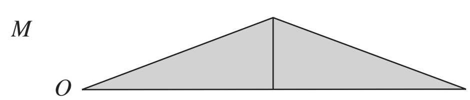
$$x_1 = \frac{11L}{32}$$

$$M_{max} = \frac{121qL^2}{2048}$$

$$M_1 = \frac{3qL^2}{64} \quad (M_{AB})_{max} = \frac{121qL^2}{2048} \quad \longleftarrow$$

LOWER BEAM

$$M_{max} = \frac{FL}{4} = \frac{5qL^2}{64}$$



$$(M_{CD})_{max} = \frac{5qL^2}{64} \quad \longleftarrow$$

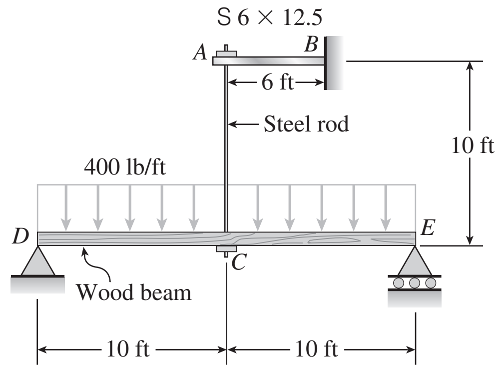
NUMERICAL VALUES

$q = 6.4$ kN/m $\quad (M_{AB})_{max} = 6.05$ kN · m $\longleftarrow$

$L = 4$ m $\quad (M_{CD})_{max} = 8.0$ kN · m $\longleftarrow$

Problem 10.4-17 The cantilever beam AB shown in the figure is an S 6 × 12.5 steel I-beam with $E = 30 \times 10^6$ psi. The simple beam DE is a wood beam 4 in. × 12 in. (nominal dimensions) in cross section with $E = 1.5 \times 10^6$ psi. A steel rod AC of diameter 0.25 in., length 10 ft, and $E = 30 \times 10^6$ psi serves as a hanger joining the two beams. The hanger fits snugly between the beams before the uniform load is applied to beam DE.

Determine the tensile force F in the hanger and the maximum bending moments M_{AB} and M_{DE} in the two beams due to the uniform load, which has intensity $q = 400$ lb/ft. (*Hint:* To aid in obtaining the maximum bending moment in beam DE, draw the shear-force and bending-moment diagrams.)

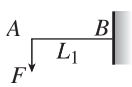


Solution 10.4-17 Beams joined by a hanger

F = tensile force in hanger

Select F as redundant.

(1) Cantilever beam AB



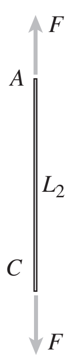
S 6 × 12.5 $I_1 = 22.1$ in.^{4}

$L_1 = 6$ ft $= 72$ in.

$E_1 = 30 \times 10^6$ psi

$$(\delta_A)_1 = \frac{FL_1^3}{3E_1I_1} = 187.66 \times 10^{-6}F \quad \begin{cases} F = \text{lb} \\ \delta = \text{in.} \end{cases}$$

(2) Hanger AC



$d = 0.25$ in. $L_2 = 10$ ft $= 120$ in.

$E_2 = 30 \times 10^6$ psi

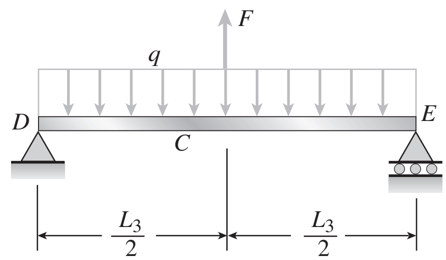
$$A_2 = \frac{\pi d^2}{4} = 0.049087 \text{ in.}^2$$

Δ = elongation of AC

$$\Delta = \frac{FL_2}{E_2A_2} = 81.488 \times 10^{-6}F$$

(F = lb, Δ = in.)

(3) Beam DCE



$L_3 = 20$ ft $= 240$ in.

$q = 400$ lb/ft

$= 33.333$ lb/in.

$E_3 = 1.5 \times 10^6$ psi

4 in. × 12 in. (nominal)

$I_3 = 415.28$ in.^{4}

$$(\delta_C)_3 = \frac{5qL_3^4}{384E_3I_3} - \frac{FL_3^3}{48E_3I_3}$$

$$= 2.3117 \text{ in.} - 462.34 \times 10^{-6} F \quad \begin{cases} F = \text{lb} \\ \delta = \text{in.} \end{cases}$$

Compatibility

$(\delta_A)_1 + \Delta = (\delta_C)_3$

$187.66 \times 10^{-6} F + 81.488 \times 10^{-6} F$

$= 2.3117 - 462.34 \times 10^{-6} F$

$F = 3160$ lb ←

(1) Max. moment in AB

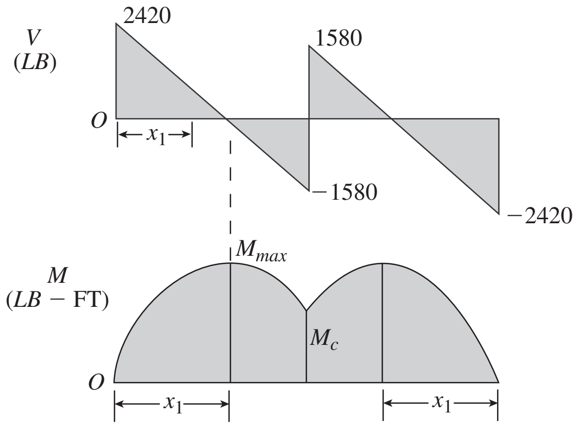
$M_{AB} = FL_1 = (3160 \text{ lb})(6 \text{ ft})$

$= 18{,}960$ lb-ft ←

(3) Max. moment in DCE

$$R_D = \frac{qL_3}{2} - \frac{F}{2} = 2420 \text{ lb}$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS



$x_1 = 6.050$ ft

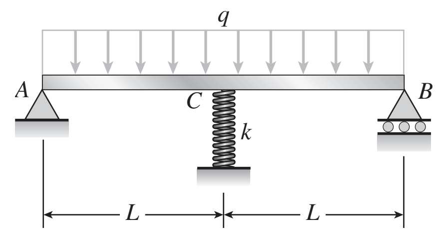
$M_{max} = 7320$ lb-ft

$M_C = 4200$ lb-ft

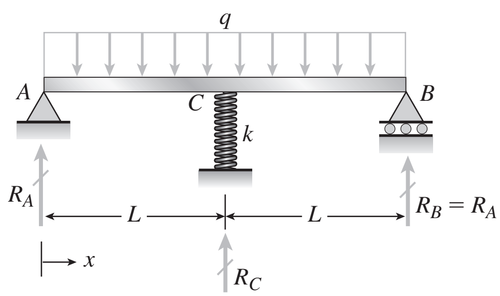
$M_{DE} = 7320$ lb-ft ⟵

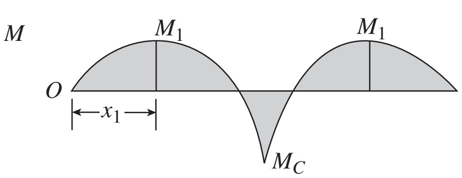
Problem 10.4-18 The beam AB shown in the figure is simply supported at A and B and supported on a spring of stiffness k at its midpoint C. The beam has flexural rigidity EI and length $2L$.

What should be the stiffness k of the spring in order that the maximum bending moment in the beam (due to the uniform load) will have the smallest possible value?



Solution 10.4-18 **Beam supported by a spring**





BENDING MOMENT $M = R_A x - \dfrac{qx^2}{2}$

LOCATION OF MAXIMUM POSITIVE MOMENT

$$\frac{dM}{dx} = 0 \quad R_A - qx = 0 \quad x_1 = \frac{R_A}{q}$$

MAXIMUM POSITIVE MOMENT

$$M_1 = (M)_{x=x_1} = \frac{R_A^2}{2q}$$

MAXIMUM NEGATIVE MOMENT

$$M_C = (M)_{x=L} = R_A L - \frac{qL^2}{2}$$

FOR THE SMALLEST MAXIMUM MOMENT:

$|M_1| = |M_C|$ or $M_1 = -M_C$

$$\frac{R_A^2}{2q} = -R_A L + \frac{qL^2}{2}$$

Solve for R_A:

$$R_A = qL(\sqrt{2} - 1)$$

EQUILIBRIUM

$$\sum F_{\text{vert}} = 0 \quad 2R_A + R_C - 2qL = 0$$

$$R_C = 2\,qL\,(2 - \sqrt{2})$$

DOWNWARD DEFLECTION OF BEAM

$$(\delta_C)_1 = \frac{5\,qL^4}{24\,EI} - \frac{R_C L^3}{6EI} = \frac{qL^4}{24\,EI}(8\sqrt{2} - 11)$$

DOWNWARD DISPLACEMENT OF SPRING

$$(\delta_C)_2 = \frac{R_C}{k} = \frac{2\,qL}{k}(2 - \sqrt{2})$$

COMPATIBILITY $(\delta_C)_1 = (\delta_C)_2$

Solve for k:

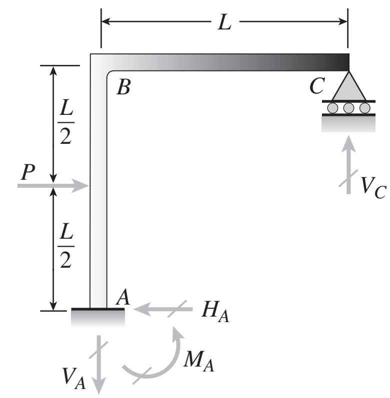
$$k = \frac{48\,EI}{7\,L^3}(6 + 5\sqrt{2})$$

$$= 89.63\frac{EI}{L^3} \quad \longleftarrow$$

Problem 10.4-19 The continuous frame *ABC* has a fixed support at *A*, a roller support at *C*, and a rigid corner connection at *B* (see figure). Members *AB* and *BC* each have length *L* and flexural rigidity *EI*. A horizontal force *P* acts at midheight of member *AB*.

(a) Find all reactions of the frame.

(b) What is the largest bending moment M_{max} in the frame? (*Note:* Disregard axial deformations in member *AB* and consider only the effects of bending.)



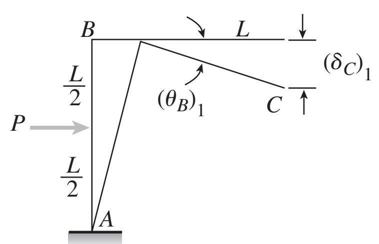
Solution 10.4-19 Frame *ABC* with fixed support

Select V_C as redundant.

EQUILIBRIUM $V_A = V_C \quad H_A = P$

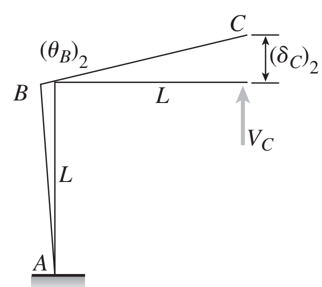
$M_A = PL/2 - V_C L$

RELEASED STRUCTURE AND FORCE-DISPL. EQS.



$$(\theta_B)_1 = \frac{PL^2}{8EI}$$

$$(\delta_C)_1 = (\theta_B)_1 L = \frac{PL^3}{8EI}$$



$$(\theta_B)_2 = \frac{V_C L^2}{EI}$$

$$(\delta_C)_2 = (\theta_B)_2 L + \frac{V_C L^3}{3EI} = \frac{4V_C L^3}{3EI}$$

COMPATIBILITY $(\delta_C)_1 = (\delta_C)_2$

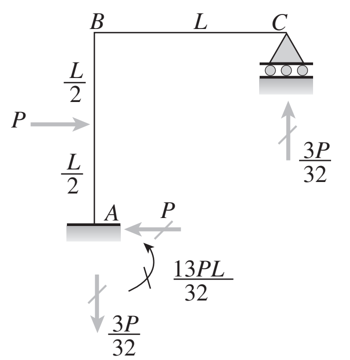
Substitute for $(\delta_C)_1$ and $(\delta_C)_2$ and solve:

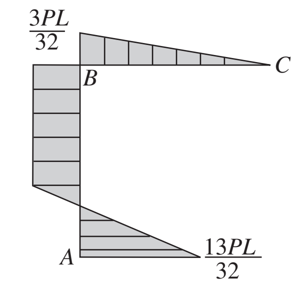
$$V_C = \frac{3P}{32} \quad \longleftarrow$$

FROM EQUILIBRIUM:

$$V_A = \frac{3P}{32} \quad H_A = P \quad M_A = \frac{13PL}{32} \quad \longleftarrow$$

REACTIONS AND BENDING MOMENTS



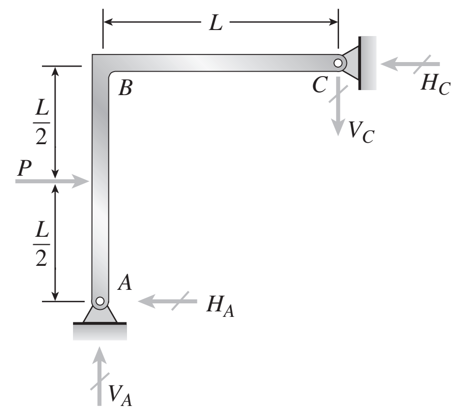


$$M_{max} = \frac{13PL}{32} \quad \longleftarrow$$

Problem 10.4-20 The continuous frame ABC has a pinned support at A, a pinned support at C, and a rigid corner connection at B (see figure). Members AB and BC each have length L and flexural rigidity EI. A horizontal force P acts at midheight of member AB.

(a) Find all reactions of the frame.

(b) What is the largest bending moment M_{max} in the frame? (*Note:* Disregard axial deformations in members AB and BC and consider only the effects of bending.)



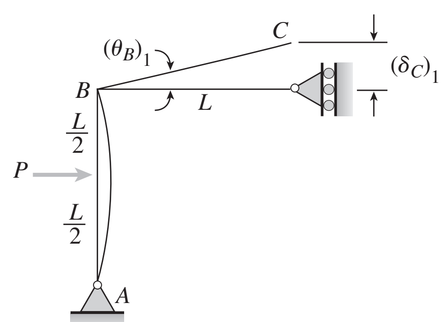
Solution 10.4-20 **Frame ABC with pinned supports**

Select V_C as redundant.

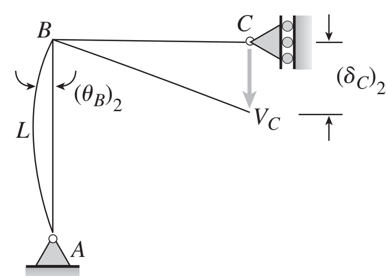
EQUILIBRIUM $V_A = V_C \quad H_A = \frac{P}{2} - V_C$

$H_C = \frac{P}{2} + V_C$

RELEASED STRUCTURE AND FORCE-DISPL. EQS.



$$(\theta_B)_1 = \frac{PL^2}{16EI} \quad (\delta_C)_1 = (\theta_B)_1 L = \frac{PL^3}{16EI}$$



$$(\theta_B)_2 = (V_C L)\frac{L}{3EI} = \frac{V_C L^2}{3EI}$$

$$(\delta_C)_2 = (\theta_B)_2 L + \frac{V_C L^3}{3EI} = \frac{2V_C L^3}{3EI}$$

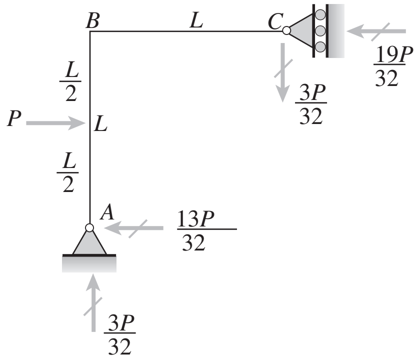
COMPATIBILITY

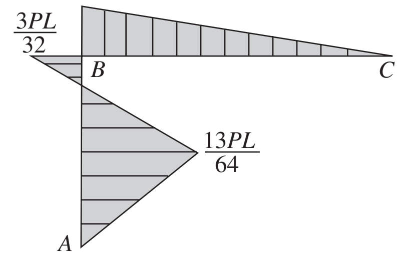
$$(\delta_C)_1 = (\delta_C)_2 \quad \frac{PL^3}{16EI} = \frac{2V_C L^3}{3EI} \quad V_C = \frac{3P}{32} \quad \longleftarrow$$

FROM EQUILIBRIUM:

$$V_A = \frac{3P}{32} \quad H_A = \frac{13P}{32} \quad H_C = \frac{19P}{32} \quad \longleftarrow$$

REACTIONS AND BENDING MOMENTS

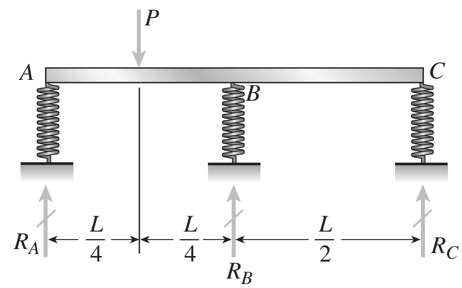




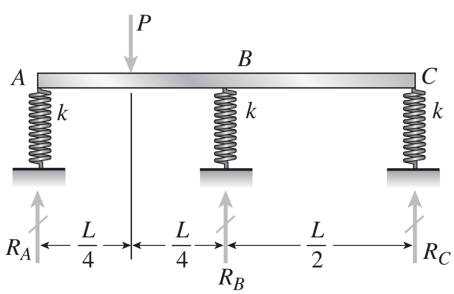
$$M_{\max} = \frac{13\,PL}{64} \longleftarrow$$

Problem 10.4-21 A wide-flange beam *ABC* rests on three identical spring supports at points *A*, *B*, and *C* (see figure). The flexural rigidity of the beam is $EI = 6912 \times 10^6$ lb-in.^{2}, and each spring has stiffness $k = 62{,}500$ lb/in. The length of the beam is $L = 16$ ft.

If the load P is 6000 lb, what are the reactions R_A, R_B, and R_C? Also, draw the shear-force and bending-moment diagrams for the beam, labeling all critical ordinates.



Solution 10.4-21 **Beam on three springs**

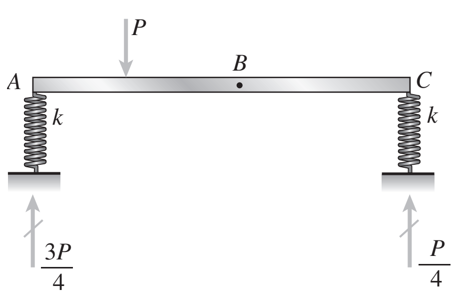


Select R_B as redundant.

EQUILIBRIUM

$$R_A = \frac{3P}{4} - \frac{R_B}{2} \qquad R_C = \frac{P}{4} - \frac{R_B}{2}$$

RELEASED STRUCTURE AND FORCE-DISPL. EQS.



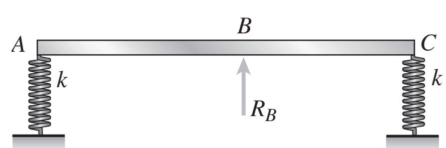
$$(\delta_A)_1 = \frac{3P}{4k}$$

$$(\delta_C)_1 = \frac{P}{4k}$$

$$(\delta_B)_1 = \frac{1}{2}[(\delta_A)_1 + (\delta_C)_1] + \frac{P\left(\frac{L}{4}\right)\left[3L^2 - 4\left(\frac{L}{4}\right)^2\right]}{48EI}$$

(Case 5, Table G-2)

$$(\delta_B)_1 = \frac{P}{2k} + \frac{11\,PL^3}{768EI} \quad \text{(downward)}$$



$$(\delta_A)_2 = \frac{R_B}{2k}$$

$$(\delta_C)_2 = \frac{R_B}{2k}$$

$$(\delta_B)_2 = \frac{1}{2}[(\delta_A)_2 + (\delta_C)_2] + \frac{R_B L^3}{48\,EI}$$

$$= \frac{R_B}{2k} + \frac{R_B L^3}{48\,EI} \quad \text{(upward)}$$

COMPATIBILITY $(\delta_B)_1 - (\delta_B)_2 = \dfrac{R_B}{k}$

Substitute and solve:

$$R_B = P\left(\frac{384\,EI + 11kL^3}{1152\,EI + 16\,kL^3}\right)$$

Let $k^* = \dfrac{kL^3}{EI}$ (nondimensional) ←

$$R_B = \frac{P}{16}\left(\frac{384 + 11k^*}{72 + k^*}\right) \quad \longleftarrow$$

FROM EQUILIBRIUM:

$$R_A = \frac{P}{32}\left(\frac{1344 + 13k^*}{72 + k^*}\right) \quad \longleftarrow$$

$$R_C = \frac{3P}{32}\left(\frac{64 - k^*}{72 + k^*}\right) \quad \longleftarrow$$

NUMERICAL VALUES

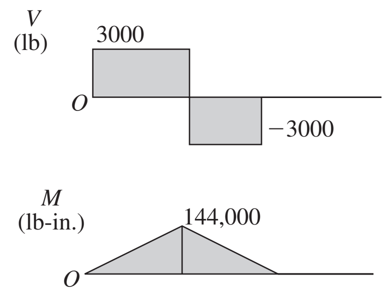
$EI = 6912 \times 10^6$ lb-in.^{2} $k = 62{,}500$ lb/in.

$L = 16$ ft $= 192$ in. $P = 6000$ lb

$k^* = \dfrac{kL^3}{EI} = 64$ $R_B = 3000$ lb ←

$R_A = 3000$ lb $R_C = 0$ ←

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS



Problem 10.4-22 A fixed-end beam AB of length L is subjected to a uniform load of intensity q acting over the middle region of the beam (see figure).

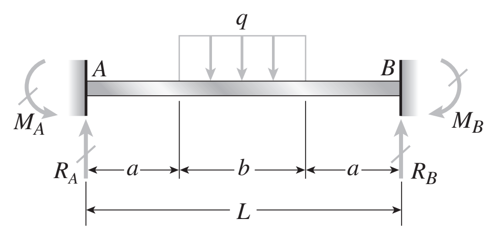
(a) Obtain a formula for the fixed-end moments M_A and M_B in terms of the load q, the length L, and the length b of the loaded part of the beam.

(b) Plot a graph of the fixed-end moment M_A versus the length b of the loaded part of the beam. For convenience, plot the graph in the following nondimensional form:

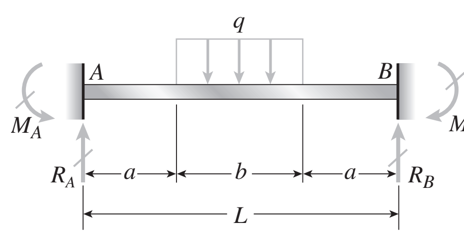
$$\frac{M_A}{qL^2/12} \quad \text{versus} \quad \frac{b}{L}$$

with the ratio b/L varying between its extreme values of 0 and 1.

(c) For the special case in which $a = b = L/3$, draw the shear-force and bending-moment diagrams for the beam, labeling all critical ordinates.



Solution 10.4-22 **Fixed-end beam**

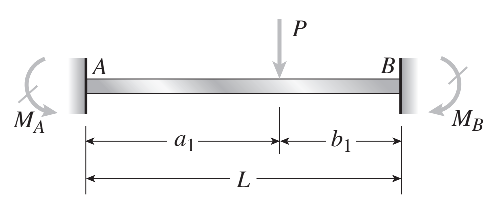


$M_A = M_B$

$$R_A = R_B = \frac{qb}{2}$$

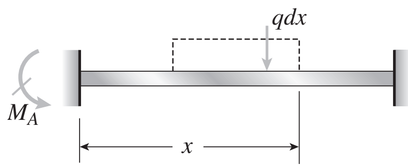
$$a = \frac{L - b}{2}$$

FROM EXAMPLE 10-4, EQ. (10-25a):



$$M_A = \frac{Pa_1b_1^2}{L^2}$$

FOR THE PARTIAL UNIFORM LOAD



$$dM_A = \frac{(qdx)(x)(L-x)^2}{L^2}$$

$$M_A = \int_a^{a+b} dM_A = \int_{(L-b)/2}^{(L+b)/2} dM_A$$

$$= \frac{q}{L^2}\int_{(L-b)/2}^{(L+b)/2} x(L-x)^2 dx$$

$$= \frac{q}{L^2}\int_{(L-b)/2}^{(L+b)/2} (L^2x - 2Lx^2 + x^3)\, dx$$

$$= \frac{q}{L^2}\left[\frac{L^2x^2}{2} - \frac{2Lx^3}{3} + \frac{x^4}{4}\right]_{(L-b)/2}^{(L+b)/2}$$

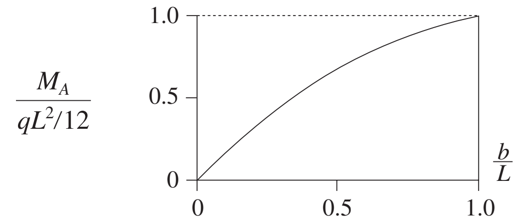
. . . (lengthy substitution) . . .

$$= \frac{qb}{24L}(3L^2 - b^2)$$

(a) $M_A = M_B = \dfrac{qb}{24L}(3L^2 - b^2)$ ←

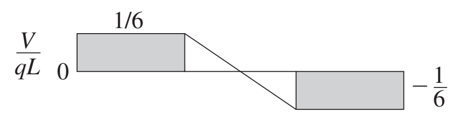
(b) GRAPH OF FIXED-END MOMENT

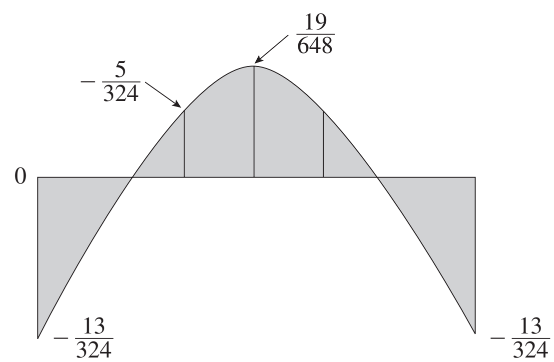
$$\frac{M_A}{qL^2/12} = \frac{b}{2L}\left(3 - \frac{b^2}{L^2}\right)$$



(c) SPECIAL CASE $a = b = L/3$

$$R_A = R_B = \frac{qL}{6} \qquad M_A = M_B = \frac{13qL^2}{324}$$



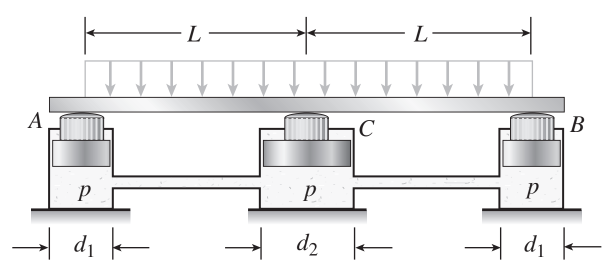


Problem 10.4-23 A beam supporting a uniform load of intensity q throughout its length rests on pistons at points A, C, and B (see figure). The cylinders are filled with oil and are connected by a tube so that the oil pressure on each piston is the same. The pistons at A and B have diameter d_1, and the piston at C has diameter d_2.

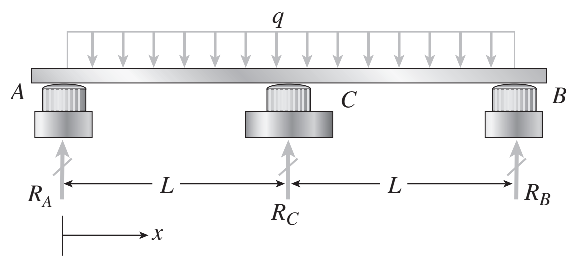
(a) Determine the ratio of d_2 to d_1 so that the largest bending moment in the beam is as small as possible.

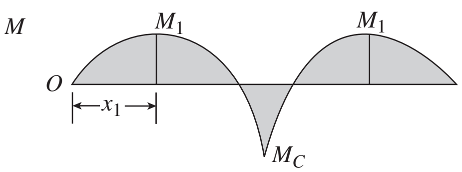
(b) Under these optimum conditions, what is the largest bending moment M_{max} in the beam?

(c) What is the difference in elevation between point C and the end supports?



Solution 10.4-23 Beam supported by pistons





BENDING MOMENT $\quad M = R_A x - \dfrac{qx^2}{2}$

LOCATION OF MAXIMUM POSITIVE MOMENT

$$\frac{dM}{dx} = 0 \qquad R_A - qx = 0 \qquad x_1 = \frac{R_A}{q}$$

MAXIMUM POSITIVE MOMENT

$$M_1 = (M)_{x=x_1} = \frac{R_A^2}{2q}$$

MAXIMUM NEGATIVE MOMENT

$$M_C = (M)_{x=L} = R_A L - \frac{qL^2}{2}$$

FOR THE SMALLEST MAXIMUM MOMENT:

$|M_1| = |M_C| \quad$ OR $\quad M_1 = -M_C$

$$\frac{R_A^2}{2q} = -R_A L + \frac{qL^2}{2}$$

Solve for R_A: $\quad R_A = qL(\sqrt{2} - 1)$

EQUILIBRIUM

$$\sum F_{\text{vert}} = 0 \qquad 2R_A + R_C - 2qL = 0$$

$$R_C = 2qL(2 - \sqrt{2})$$

REACTIONS BASED UPON PRESSURE

$$R_A = R_B = p\left(\frac{\pi d_1^2}{4}\right) \qquad R_C = p\left(\frac{\pi d_2^2}{4}\right)$$

(a) $\quad \therefore \dfrac{d_2}{d_1} = \sqrt{\dfrac{R_C}{R_A}} = \sqrt{\dfrac{2(2-\sqrt{2})}{\sqrt{2}-1}} = \sqrt[4]{8}$

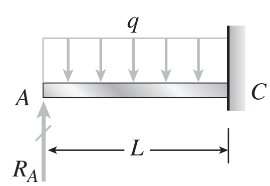
$= 1.682 \longleftarrow$

(b) $\quad M_{\text{MAX}} = M_1 = \dfrac{R_A^2}{2q} = \dfrac{qL^2}{2}(3 - 2\sqrt{2})$

$= 0.08579\, qL^2 \longleftarrow$

(c) DIFFERENCE IN ELEVATION

By symmetry, beam has zero slope at C.



$R_A = qL(\sqrt{2} - 1)$

δ_A = Difference in elev.

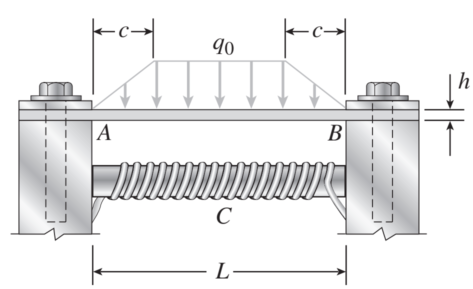
$$\delta_A = \frac{R_A L^3}{3EI} - \frac{qL^4}{8EI} = \frac{qL^4}{24EI}(8\sqrt{2} - 11)$$

$= 0.01307\, qL^4/EI \longleftarrow$

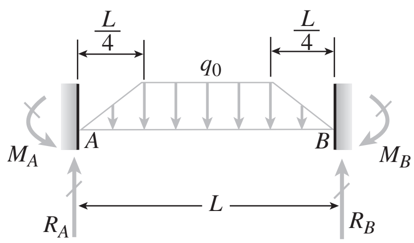
Point C is below points A and B by the amount $0.01307qL^4/EI$.

Problem 10.4-24 A thin steel beam AB used in conjunction with an electromagnet in a high-energy physics experiment is securely bolted to rigid supports (see figure). A magnetic field produced by coils C results in a force acting on the beam. The force is trapezoidally distributed with maximum intensity $q_0 = 18$ kN/m. The length of the beam between supports is $L = 200$ mm and the dimension c of the trapezoidal load is 50 mm. The beam has a rectangular cross section with width $b = 60$ mm and height $h = 20$ mm.

Determine the maximum bending stress $\sigma_{\max}$ and the maximum deflection $\delta_{\max}$ for the beam. (Disregard any effects of axial deformations and consider only the effects of bending. Use $E = 200$ GPa.)



Solution 10.4-24 Fixed-end beam (trapezoidal load)



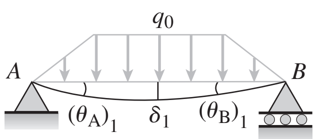
$c = \dfrac{L}{4}$

FROM SYMMETRY AND EQUILIBRIUM

$$M_A = M_B \qquad R_A = R_B = \frac{3q_0L}{8}$$

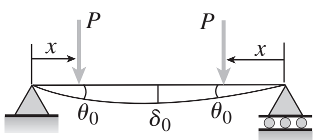
SELECT M_A AND M_B AS REDUNDANTS

RELEASED STRUCTURE WITH APPLIED LOAD



$(\theta_A)_1 = (\theta_B)_1$

Consider the following beam from Case 6, Table G-2:



$$\theta_0 = \frac{Px(L-x)}{2EI} \qquad \delta_0 = \frac{Px}{24EI}(3L^2 - 4x^2)$$

Consider the load P as an element of the distributed load.

Replace P by qdx, where

$$q = \frac{4q_0x}{L} \qquad x \text{ from 0 to } L/4$$

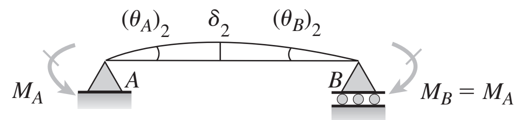
$$q = q_0 \qquad x \text{ from } L/4 \text{ to } L/2$$

$$(\theta_A)_1 = \frac{1}{2EI}\int_0^{L/4}\left(\frac{4q_0x}{L}\right)(x)(L-x)\,dx + \frac{1}{2EI}\int_{L/4}^{L/2} q_0x(L-x)\,dx$$

$$= \frac{13q_0L^3}{1536\,EI} + \frac{11q_0L^3}{384\,EI} = \frac{19q_0L^3}{512EI}$$

$$\delta_1 = \frac{1}{24EI}\int_0^{L/4}\left(\frac{4q_0x}{L}\right)(x)(3L^2-4x^2)\,dx + \frac{1}{24EI}\int_{L/4}^{L/2} q_0x(3L^2-4x^2)\,dx$$

$$= \frac{19q_0L^4}{7680EI} + \frac{19q_0L^4}{2048EI} = \frac{361q_0L^4}{30{,}720EI}$$



RELEASED STRUCTURE WITH REDUNDANTS

$$(\theta_A)_2 = (\theta_B)_2 \qquad M_B = M_A$$

FROM Case 10, Table G-2:

$$(\theta_A)_2 = \frac{M_AL}{2EI} \qquad \delta_2 = \frac{M_AL^2}{8EI}$$

COMPATIBILITY

$$\theta_A = (\theta_A)_1 - (\theta_A)_2 = 0$$

$$\frac{19\,q_0L^3}{512\,EI} - \frac{M_AL}{2\,EI} = 0 \qquad M_A = \frac{19q_0L^2}{256}$$

DEFLECTION AT THE MIDPOINT

$$\delta_{max} = \delta_1 - \delta_2 = \frac{361q_0L^4}{30{,}720\,EI} - \frac{M_AL^2}{8\,EI}$$

$$= \frac{361q_0L^4}{30{,}720\,EI} - \left(\frac{19q_0L^2}{256}\right)\left(\frac{L^2}{8\,EI}\right)$$

$$= \frac{19q_0L^4}{7680\,EI}$$

BENDING MOMENT AT THE MIDPOINT

$$M_C = R_A\left(\frac{L}{2}\right) - M_A - \frac{q_0L^2}{24} - \frac{q_0L^2}{32}$$

$$= \frac{3q_0L}{8}\left(\frac{L}{2}\right) - \frac{19q_0L^2}{256} - \frac{7q_0L^2}{96} = \frac{31q_0L^2}{768}$$

MAXIMUM BENDING MOMENT

$$M_A > M_C \qquad \therefore\ M_{max} = M_A = \frac{19q_0L^2}{256}$$

NUMERICAL VALUES

$q_0 = 18$ kN/m $\quad L = 200$ mm $\quad b = 60$ mm

$h = 20$ mm $\quad E = 200$ GPa

$$s = \frac{bh^2}{6} = 4.0 \times 10^{-6}\ \text{m}^3$$

$$I = \frac{bh^3}{12} = 40 \times 10^{-9}\ \text{m}^4$$

$$M_{max} = \frac{19\, q_0 L^2}{256} = 53.44\ \text{N} \cdot \text{m}$$

$$\sigma_{max} = \frac{M_{max}}{S} = 13.4\ \text{MPa} \longleftarrow$$

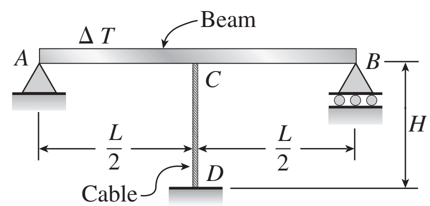
$$\delta_{max} = \frac{19\, q_0 L^4}{7680\, EI} = 0.00891\ \text{mm} \longleftarrow$$

Temperature Effects

The beams described in the problems for Section 10.5 have constant flexural rigidity EI.

Problem 10.5-1 A cable CD of length H is attached to the midpoint of a simple beam AB of length L (see figure). The moment of inertia of the beam is I, and the effective cross-sectional area of the cable is A. The cable is initially taut but without any initial tension.

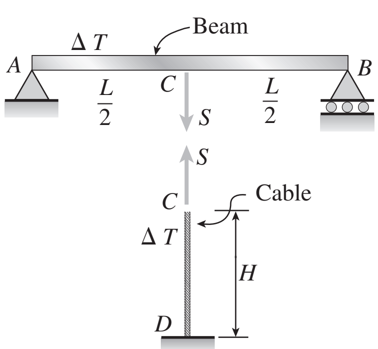
Obtain a formula for the tensile force S in the cable when the temperature drops uniformly by ΔT degrees, assuming that the beam and cable are made of the same material (modulus of elasticity E and coefficient of thermal expansion α). (Use the method of superposition in the solution.)



Solution 10.5-1 **Uniform temperature change**

ΔT = Decrease in temperature use method of superposition. Select tensile force S in the cable as redundant.

RELEASED STRUCTURE



I = Moment of inertia

A = Cross-sectional area

BEAM $\quad (\delta_C)_1 = \dfrac{SL^3}{48\,EI}$ (downward)

CABLE $\quad (\delta_C)_2 = \alpha H(\Delta T) - \dfrac{SH}{EA}$ (downward)

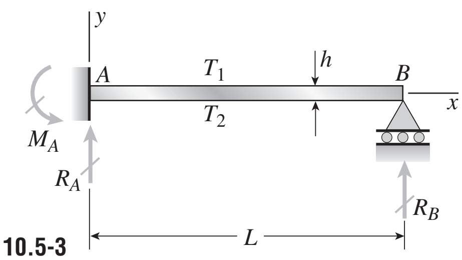
COMPATIBILITY $\quad (\delta_C)_1 = (\delta_C)_2$

$$\frac{SL^3}{48\,EI} = \alpha H(\Delta T) - \frac{SH}{EA}$$

SOLVE FOR S: $\quad S = \dfrac{48\,EIAH\alpha(\Delta T)}{AL^3 + 48\,IH} \longleftarrow$

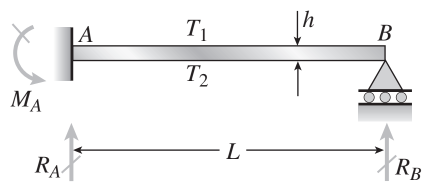
Problem 10.5-2 A propped cantilever beam, fixed at the left-hand end A and simply supported at the right-hand end B, is subjected to a temperature differential with temperature T_1 on its upper surface and T_2 on its lower surface (see figure).

Find all reactions for this beam. (Use the method of superposition in the solution. Also, if desired, use the results from Problem 9.13-1.)



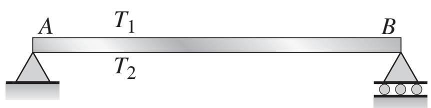
Probs. 10.5-2 and 10.5-3

Solution 10.5-2 **Beam with temperature differential**



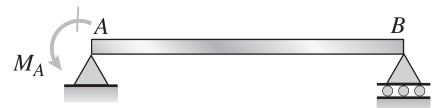
Use the method of superposition.
Select M_A as redundant.

RELEASED STRUCTURE



$$(\theta_A)_1 = \frac{\alpha L(T_2 - T_1)}{2h} \quad \text{(clockwise)}$$

(From the answer to prob. 9.11-1)



$$(\theta_A)_2 = \frac{M_A L}{3EI} \quad \text{(counterclockwise)}$$

COMPATIBILITY $(\theta_A)_1 = (\theta_A)_2$

$$\frac{\alpha L(T_2 - T_1)}{2h} = \frac{M_A L}{3EI} \qquad M_A = \frac{3\alpha EI(T_2 - T_1)}{2h} \longleftarrow$$

EQUILIBRIUM

$$\sum M_B = 0 \qquad M_A - R_A L = 0$$

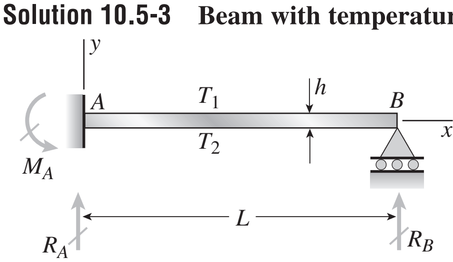
$$R_A = \frac{3\alpha EI(T_2 - T_1)}{2hL} \longleftarrow$$

$$\sum F_{\text{vert}} = 0 \qquad R_B = -R_A$$

$$R_B = -\frac{3\alpha EI(T_2 - T_1)}{2hL} \longleftarrow$$

Problem 10.5-3 Solve the preceding problem by integrating the differential equation of the deflection curve.

Solution 10.5-3 **Beam with temperature differential**



$$M = R_B(L - x)$$

DIFFERENTIAL EQUATION (EQ. 10-39b)

$$EIv'' = M + \frac{\alpha EI(T_2 - T_1)}{h}$$

$$\text{or} \quad EIv'' = R_B(L - x) + \frac{\alpha EI(T_2 - T_1)}{h}$$

$$EIv' = R_B Lx - R_B\left(\frac{x^2}{2}\right) + \frac{\alpha EI(T_2 - T_1)}{h}x + C_1$$

B.C. 1 $\quad v'(0) = 0 \qquad \therefore C_1 = 0$

$$EIv = R_B L\left(\frac{x^2}{2}\right) - R_B\left(\frac{x^3}{6}\right) + \frac{\alpha EI(T_2 - T_1)}{2h}x^2 + C_2$$

B.C. 2 $\quad v(0) = 0 \qquad \therefore C_2 = 0$

B.C. 3 $\quad v(L) = 0$

$$\therefore\ R_B = -\frac{3\alpha EI\,(T_2 - T_1)}{2hL} \longleftarrow$$

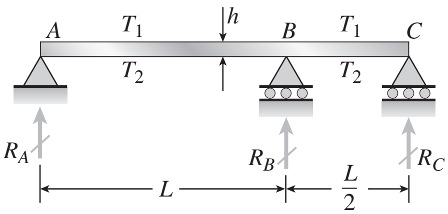
FROM EQUILIBRIUM:

$$R_A = -R_B = \frac{3\alpha EI(T_2 - T_1)}{2hL} \longleftarrow$$

$$M_A = R_A L \qquad M_A = \frac{3\alpha EI(T_2 - T_1)}{2h} \longleftarrow$$

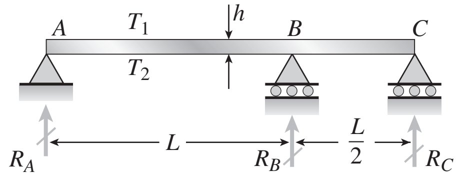
Problem 10.5-4 A two-span beam with spans of lengths L and $L/2$ is subjected to a temperature differential with temperature T_1 on its upper surface and T_2 on its lower surface (see figure).

Determine all reactions for this beam. (Use the method of superposition in the solution. Also, if desired, use the results from Problems 9.8-5 and 9.13-3.)



Probs. 10.5-4 and 10.5-5

Solution 10.5-4 Beam with temperature differential

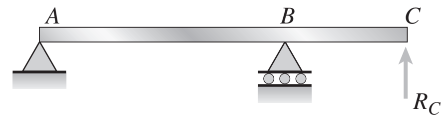


Use the method of superposition.
Select R_C as redundant.

RELEASED STRUCTURE

From prob. 9.13-3:

$$(\delta_C)_1 = \frac{3\alpha L^2 (T_2 - T_1)}{8h} \text{ (upward)}$$



From prob. 9.8-5:

$$(\delta_C)_2 = \frac{R_C L^3}{8EI} \text{ (upward)} \quad \leftarrow$$

COMPATIBILITY $(\delta_C)_1 + (\delta_C)_2 = 0$

$$\frac{3\alpha L^2(T_2 - T_1)}{8h} = -\frac{R_C L^3}{8EI}$$

$$R_C = -\frac{3\alpha EI(T_2 - T_1)}{hL} \quad \leftarrow$$

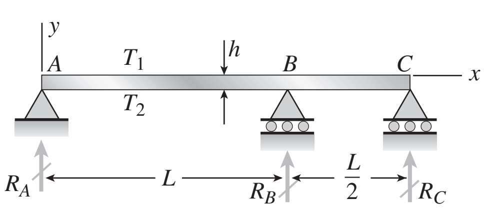
FROM EQUILIBRIUM:

$$R_A = \frac{R_C}{2} \qquad R_A = -\frac{3\alpha EI(T_2 - T_1)}{2hL} \quad \leftarrow$$

$$R_B = -\frac{3R_C}{2} \qquad R_B = \frac{9\alpha EI(T_2 - T_1)}{2hL} \quad \leftarrow$$

Problem 10.5-5 Solve the preceding problem by integrating the differential equation of the deflection curve.

Solution 10.5-5 Beam with temperature differential



DIFFERENTIAL EQUATION (EQ. 10-39b)

$$EIv'' = M + \frac{\alpha EI(T_2 - T_1)}{h}$$

For convenience, Let $\beta = \dfrac{\alpha EI(T_2 - T_1)}{h}$ (1)

$EIv'' = M + \beta$ (2)

PART AB OF THE BEAM $(0 \le x \le L)$

$M = R_A x \qquad EIv'' = R_A x + \beta$

$EIv' = R_A x^2/2 + \beta x + C_1$ (3)

$EIv = R_A x^3/6 + \beta x^2/2 + C_1 x + C_2$ (4)

B.C. 1 $\quad v(0) = 0 \quad \therefore C_2 = 0$

B.C. 2 $\quad v(L) = 0 \quad \therefore R_A L^2 + 6C_1 = -3\beta L$ (5)

PART BC OF THE BEAM ($L \le x \le 3L/2$)

$M = R_A x + R_B(x - L)$

From equilibrium, $R_B = -3R_A$ (6)

$\therefore M = -2R_A x + 3R_A L$

$EIv'' = M + \beta = -2R_A x + 3R_A L + \beta$

$EIv' = -R_A x^2 + 3R_A Lx + \beta x + C_3$ (7)

$EIv = -R_A x^3/3 + 3R_A Lx^2/2 + \beta x^2/2 + C_3 x + C_4$

B.C. 3 $v(L) = 0$ (8)

$\therefore 7R_A L^3 + 6C_3 L + 6C_4 = -3\beta L^2$ (9)

B.C. 4 $v(3L/2) = 0$

$\therefore 18R_A L^3 + 12C_3 L + 8C_4 = -9\beta L^2$ (10)

CONTINUITY CONDITION AT B

$(EIv')_{AB} = (EIv')_{BC}$ At $x = L$

From Eqs. (3) and (7):

$R_A(L^2/2) + \beta L + C_1 = -R_A L^2 + 3R_A L^2 + \beta L + C_3$

or $3R_A L^2 - 2C_1 + 2C_3 = 0$ (11)

SOLVE EQS. (5), (9), (10), AND (11) FOR R_A:

$$R_A = -\frac{3\beta}{2L} = -\frac{3\alpha EI(T_2 - T_1)}{2hL} \longleftarrow$$

Also: $C_1 = -\beta L/4$ $C_2 = 0$ $C_3 = 2\beta L$

$C_4 = -3\beta L^2/4$

From Eq. (6): $R_B = \dfrac{9\alpha EI(T_2 - T_1)}{2hL} \longleftarrow$

From equilibrium:

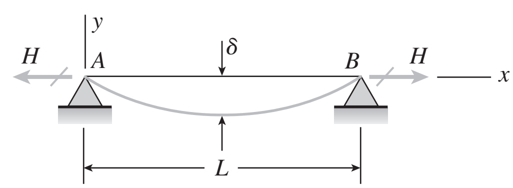
$$R_C = 2R_A = -\frac{3\alpha EI(T_2 - T_1)}{hL} \longleftarrow$$

Longitudinal Displacements at the Ends of Beams

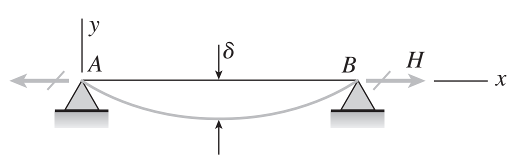
Problem 10.6-1 Assume that the deflected shape of a beam AB with *immovable* pinned supports (see figure) is given by the equation $v = -\delta \sin \pi x/L$, where δ is the deflection at the midpoint of the beam and L is the length. Also, assume that the beam has constant axial rigidity EA.

(a) Obtain formulas for the longitudinal force H at the ends of the beam and the corresponding axial tensile stress σ_t.

(b) For an aluminum-alloy beam with $E = 10 \times 10^6$ psi, calculate the tensile stress σ_t when the ratio of the deflection δ to the length L equals 1/200, 1/400, and 1/600.



Solution 10.6-1 **Beam with immovable supports**



(a)

$$v = -\delta \sin\frac{\pi x}{L} \qquad \frac{dv}{dx} = -\frac{\pi\delta}{L}\cos\frac{\pi x}{L}$$

Eq. (10-42): $\lambda = \dfrac{1}{2}\displaystyle\int_0^L \left(\frac{dv}{dx}\right)^2 dx = \frac{\pi^2\delta^2}{4L}$

Eq. (10-45): $H = \dfrac{EA\lambda}{L} = \dfrac{\pi^2 EA\delta^2}{4L^2} \longleftarrow$

Eq. (10-46): $\sigma_t = \dfrac{H}{A} = \dfrac{\pi^2 E\delta^2}{4L^2} \longleftarrow$

(b) ALUMINUM ALLOY

$$E = 10 \times 10^6 \text{ psi} \qquad \sigma_t = 24.67 \times 10^6 \left(\frac{\delta}{L}\right)^2 \text{(psi)}$$

$\frac{\delta}{L}$	$\frac{1}{200}$	$\frac{1}{400}$	$\frac{1}{600}$
σ_t (psi)	617	154	69

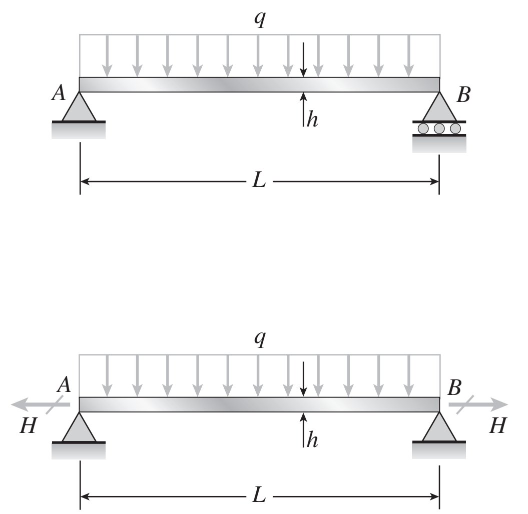
Note: The axial stress increases as the deflection increases.

Problem 10.6-2 (a) A simple beam AB with length L and height h supports a uniform load of intensity q (see the *first part* of the figure). Obtain a formula for the curvature shortening λ of this beam. Also, obtain a formula for the maximum bending stress σ_b in the beam due to the load q.

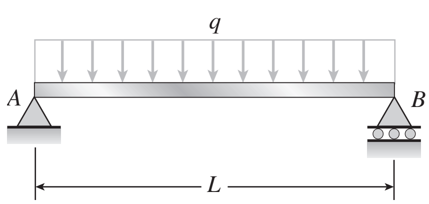
(b) Now assume that the ends of the beam are pinned so that curvature shortening is prevented and a horizontal force H develops at the supports (see the *second part* of the figure). Obtain a formula for the corresponding axial tensile stress σ_t.

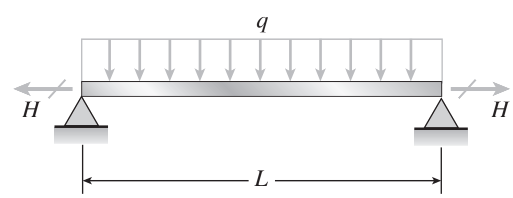
(c) Using the formulas obtained in parts (a) and (b), calculate the curvature shortening λ, the maximum bending stress σ_b, and the tensile stress σ_t for the following steel beam: length $L = 3$ m, height $h = 300$ mm, modulus of elasticity $E = 200$ GPa, and moment of inertia $I = 36 \times 10^6$ mm^4. Also, the load on the beam has intensity $q = 25$ kN/m.

Compare the tensile stress σ_t produced by the axial forces with the maximum bending stress σ_b produced by the uniform load.



Solution 10.6-2 Beam with uniform load





h = Height of beam

(a) CURVATURE SHORTENING

From Case 1, Table G-2:

$$\frac{dv}{dx} = -\frac{q}{24EI}(L^3 - 6Lx^2 - 4x^3)$$

$$\text{Eq. (10-42): } \lambda = \frac{1}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx$$

$$= \frac{17q^2L^7}{40{,}320\,E^2I^2} \quad \longleftarrow$$

BENDING STRESS

$$M_{max} = \frac{qL^2}{8} \qquad c = \frac{h}{2}$$

$$\sigma_b = \frac{M_c}{I} = \frac{qhL^2}{16I} \quad \longleftarrow$$

(b) IMMOVABLE SUPPORTS

$$\text{Eq. (10-45): } H = \frac{EA\lambda}{L}$$

$$\text{Eq. (10-46): } \sigma_t = \frac{H}{A} = \frac{E\lambda}{L} = \frac{17q^2L^6}{40{,}320\,EI^2} \quad \longleftarrow$$

(c) NUMERICAL VALUES $q = 25$ kN/m

$L = 3$ m $\quad h = 300$ mm $\quad E = 200$ GPa

$I = 36 \times 10^6$ mm^4 $\quad \lambda = 0.01112$ mm $\longleftarrow$

$\sigma_b = 117.2$ MPa $\quad \sigma_t = 0.7411$ MPa $\longleftarrow$

The bending stress is much larger than the axial tensile stress due to curvature shortening.